LEFT-ORDERABLE, NON–L–SPACE SURGERIES ON KNOTS

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In honor of Dale Rolfsen

Abstract. Let $K$ be a knot in the 3-sphere $S^3$. An $r$–surgery on $K$ is left-orderable if the resulting 3–manifold $K(r)$ of the surgery has left-orderable fundamental group, and an $r$–surgery on $K$ is called an $L$–space surgery if $K(r)$ is an $L$–space. A conjecture of Boyer, Gordon and Watson says that non-reducing surgeries on $K$ can be classified into left-orderable surgeries or $L$–space surgeries. We introduce a way to provide knots with left-orderable, non–$L$–space surgeries. As an application we present infinitely many hyperbolic knots on each of which every nontrivial surgery is a hyperbolic, left-orderable, non–$L$–space surgery.

1. Introduction

A nontrivial group $G$ is said to be left-orderable if there exists a strict total ordering $<$ on its elements such that $g < h$ implies $fg < fh$ for all elements $f, g, h \in G$. The left-orderability of fundamental groups of 3–manifolds has been studied by Boyer, Rolfsen and Wiest [5]. In particular, they prove that the fundamental group of a $P^2$–irreducible 3–manifold is left-orderable if and only if it has an epimorphism to a left-orderable group [5, Theorem 1.1(1)]. Since the infinite cyclic group $\mathbb{Z}$ is left-orderable, a $P^2$–irreducible 3–manifold with first Betti number $b_1 \geq 1$ has left-orderable fundamental group. One obstruction for $G$ being left-orderable is an existence of torsion elements in $G$. Thus, for instance, lens spaces cannot have left-orderable fundamental groups. It is interesting to characterize rational homology 3–spheres whose fundamental groups are left-orderable. Examples suggest that there exists a correspondence between rational homology 3–spheres whose fundamental groups cannot be left-ordered and $L$–spaces which appear in the Heegaard Floer homology theory [45, 46]. For a rational homology 3–sphere $M$, we have $\text{rk}\tilde{HF}(M) \geq |H_1(M;\mathbb{Z})|$. If the equality holds, then $M$ is called an $L$–space. Following [4, 1.1], for homogeneity, we use $\mathbb{Z}_2$-coefficients for Heegaard Floer homology.
The present paper is motivated by the following conjecture formulated by Boyer, Gordon and Watson [4].

**Conjecture 1.1** ([4]). An irreducible rational homology 3–sphere is an $L$–space if and only if its fundamental group is not left-orderable.

In [4] the conjecture is verified for geometric, non-hyperbolic 3–manifolds and the 2–fold branched covers of non-splitting alternating links. See also [2, 8, 19, 27, 49] for related results.

A useful way to construct rational homology 3–spheres is Dehn surgery on knots in the 3–sphere $S^3$. Henceforth we will focus on Conjecture 1.1 for rational homology 3–spheres obtained by Dehn surgery on knots in $S^3$. For any knot $K$ in $S^3$ the exterior $E(K) = S^3 - \text{int} N(K)$ has left-orderable fundamental group [5, Corollary 3.5]. However, the result $K(r)$ of $r$–Dehn surgery may not have such a fundamental group; see Examples 1.6, [11] and [30].

A Dehn surgery is said to be **left-orderable** if the resulting 3–manifold of the surgery has left-orderable fundamental group. Define the set of left-orderable surgeries on $K$ as

$$S_{LO}(K) = \{ r \in \mathbb{Q} \mid \pi_1(K(r)) \text{ is left-orderable} \}.$$

Similarly a Dehn surgery is called an **$L$–space surgery** if the resulting 3–manifold of the surgery is an $L$–space, and the set of $L$–space surgeries on $K$ is defined as

$$S_L(K) = \{ r \in \mathbb{Q} \mid K(r) \text{ is an } L\text{–space} \}.$$

**Remark 1.2.**

1. Note that 0–surgery does not yield a rational homology 3–sphere, and hence $K(0)$ is not an $L$–space and $0 \notin S_L(K)$. On the other hand, if $K$ is a trivial knot, then $K(0) \cong S^2 \times S^1$ which has left orderable fundamental group. If $K$ is a nontrivial knot, then $K(0)$ is irreducible [15, Corollary 8.3] and $H_1(K(0)) \cong \mathbb{Z}$, hence $0 \in S_{LO}(K)$ [5, Theorem 1.1].

2. Let $K^*$ be the mirror image of a knot $K$, and put $-S = \{ -r \mid r \in S \}$ for $S \subset \mathbb{Q}$. Since $K^*(-r)$ is orientation reversingly diffeomorphic to $K(r)$ and the conditions of a 3–manifold $M$ having left-orderable fundamental group and being an $L$–space are independent of the orientation of $M$ [47, p.1288], we have $S_{LO}(K^*) = -S_{LO}(K)$ and $S_L(K^*) = -S_L(K)$.

If $K(r)$ is a reducible 3–manifold for a nontrivial knot $K$, it has a lens space summand [18, Theorem 3], hence $r \notin S_{LO}(K)$, but $r$ may or may not be in $S_L(K)$; see Remark 1.4 and Example 1.6.

If $K(r)$ is irreducible, Conjecture 1.1 asserts that $r$ belongs to exactly one of $S_{LO}(K)$ and $S_L(K)$. Taking the cabling conjecture [16] into consideration, Conjecture 1.1 suggests:
Conjecture 1.3. Let $K$ be a knot in $S^3$ which is not a cable of a nontrivial knot. Then $S_{\text{LO}}(K) \cup S_L(K) = \mathbb{Q}$ and $S_{\text{LO}}(K) \cap S_L(K) = \emptyset$.

Remark 1.4. The cabling conjecture [16] asserts that if $K(r)$ is reducible for a nontrivial knot $K$, then $K$ is cabled and $r$ is a cabling slope. There exists a cable knot $K$ for which $S_{\text{LO}}(K) \cup S_L(K) \neq \mathbb{Q}$. For instance, let $K$ be a $(p,q)$ cable of a non-fibered knot $(q > 0)$. Then $K(pq) = k(q)\mathbb{L}(q,p)$ [17, Corollary 7.3]. Since $\pi_1(K(pq))$ has a torsion, $pq \notin S_{\text{LO}}(K)$. To see that $pq \notin S_L(K)$, we note that $\widehat{HF}(K(pq)) \cong \widehat{HF}(k(q)) \otimes \widehat{HF}(\mathbb{L}(q,p))$; see [53, 8.1(5)] ([46]). Since $k$ is a non-fibered knot, $k(q)$ is not an $L$–space [43, 44]. Hence the rank of $\widehat{HF}(K(pq))$ is strictly bigger than $|p|q$, and $K(pq)$ is not an $L$–space. It follows that $pq \notin S_{\text{LO}}(K) \cup S_L(K)$.

For the trivial knot and nontrivial torus knots, Examples 1.5 and 1.6 describe $S_{\text{LO}}(K)$ and $S_L(K)$ explicitly. Note that these knots satisfy Conjecture 1.3.

Example 1.5 (trivial knot). Let $K$ be the trivial knot in $S^3$. Then $S_{\text{LO}}(K) = \{0\}$ and $S_L(K) = \mathbb{Q} - \{0\}$.

Example 1.6 (torus knots). For a nontrivial torus knot $T_{p,q}$ ($p > q \geq 2$), the argument in the proof of [10, Theorem 1.4] shows that $S_{\text{LO}}(T_{p,q}) = (-\infty, pq - p - q) \cap \mathbb{Q}$ and $S_L(T_{p,q}) = [pq - p - q, \infty) \cap \mathbb{Q}$.

Example 1.7 (figure-eight knot). Let $K$ be the figure-eight knot. Following [47, 48], $S_L(K) = \emptyset$. Thus it is expected that $S_{\text{LO}}(K) = \mathbb{Q}$. Boyer, Gordon and Watson [4] show that $S_{\text{LO}}(K) \supset (-4,4) \cap \mathbb{Q}$, and Clay, Lidman and Watson [8] improve that $S_{\text{LO}}(K) \supset [-4,4] \cap \mathbb{Q}$. Furthermore, [14] implies that $S_{\text{LO}}(K) \supset \mathbb{Z}$.

For related results, see [9, 11, 20, 33, 54, 56].

It is known that there exist some constraints for knots which admit $L$–space surgeries. For instance, such knots have specific Alexander polynomials [47], and must be fibered [43, 44]. Thus generically we have $S_L(K) = \emptyset$. Hence Conjecture 1.3 suggests that $S_{\text{LO}}(K) = \mathbb{Q}$ for most knots. Despite being expected, there is no literature giving explicitly knots with $S_{\text{LO}}(K) = \mathbb{Q}$ and $S_L(K) = \emptyset$. In the present note we give infinitely many satellite knots and hyperbolic knots with this property.

Theorem 1.8. Given a nontrivial knot $K'$, there are infinitely many prime satellite knots $K$ each of which has $K'$ as a companion knot and enjoys the following properties:

(1) $K(r)$ is a toroidal 3–manifold which is not a graph manifold for all but finitely many $r \in \mathbb{Q}$.

(2) $S_{\text{LO}}(K) = \mathbb{Q}$.
This is an application of Proposition 7.1 due to Clay and Watson [10, Proposition 4.1]. In Theorem 1.8, $K$ is satellite knot and the resulting 3–manifold $K(r)$ has a nontrivial Jaco-Shalen-Johannson (JSJ) decomposition [28, 29]. Since Proposition 7.1 does not work for creation of hyperbolic knots, we will introduce an effective way to provide infinitely many hyperbolic knots having left-orderable, non–$L$–space surgeries from a given knot with left-orderable surgeries; see Section 4. Then we will apply the construction to prove the following:

**Theorem 1.9.** There exist infinitely many hyperbolic knots $K$ each of which enjoys the following properties.

1. $K(r)$ is a hyperbolic 3–manifold for all $r \in \mathbb{Q}$.
2. $\mathcal{S}_{LO}(K) = \mathbb{Q}$.
3. $\mathcal{S}_{L}(K) = \emptyset$.

## 2. Left-orderable surgeries on periodic knots

A knot $K$ in $S^3$ is called a *periodic knot* with period $p$ if there is an orientation preserving diffeomorphism $f : S^3 \to S^3$ such that $f(K) = K$, $f^p = id$ ($p > 1$), $\text{Fix}(f) \neq \emptyset$, and $\text{Fix}(f) \cap K = \emptyset$, where $\text{Fix}(f)$ is the set of fixed points of $f$. By the positive answer to the Smith conjecture [40], $f$ is a rotation of $S^3$ about the unknotted circle $C = \text{Fix}(f)$. So by taking the quotient $S^3/(f)$, we obtain the *factor knot* $\mathcal{K} = K/(f)$ and the unknotted circle $\mathcal{C} = C/(f)$ in $S^3 = S^3/(f)$. We often call $C$ the *axis* and $\mathcal{C}$ the *branch circle*. Since $K$ is connected, the linking number $\text{lk}(\mathcal{K}, \mathcal{C})$ and the period $p$ are relatively prime. Note that if the periodic knot $K$ is unknotted, then the equivariant loop theorem [36] implies that $K \cup C$ is the Hopf link and $\mathcal{K} \cup \mathcal{C}$ is also the Hopf link. To exclude such a trivial case, in the following we consider nontrivial periodic knots.

![Figure 2.1](image-url) **Figure 2.1.** A periodic knot $K$ with an axis $C$ and its factor knot $\mathcal{K}$; $T$ is a tangle.
The branched cover $V$; Let $K$ be a nontrivial periodic knot, the filled solid torus where $V$; to $p$ giving the cyclic period $p$ and a positive integer $p$. Note that if $S = \mathbb{Q}$, then $pS = \mathbb{Q}$.

**Theorem 2.1.** Let $K$ be a nontrivial knot in $S^3$ with cyclic period $p$, and let $\overline{K}$ be its factor knot. Then $S_{LO}(K) \supset pS_{LO}(\overline{K})$.

**Proof of Theorem 2.1.** Let $f : S^3 \to S^3$ be an orientation preserving diffeomorphism giving the cyclic period $p$ of $K$ with axis $C = \text{Fix}(f)$ and factor knot $\overline{K} = K/(f)$. Take an $(f)$-invariant tubular neighborhood $N(K)$ of $K$. Let $N(\overline{K})$ be the quotient $N(K)/(f)$. In the following $E(K) = S^3 - \text{int}(N(K))$ and $E(\overline{K}) = E(K)/(f) = S^3 - \text{int}(N(\overline{K}))$. Denote by $(\mu, \lambda)$ (resp. $(\bar{\mu}, \bar{\lambda})$) a preferred meridian-longitude pair of $\pi_1(\partial N(K))$ (resp. $\pi_1(\partial N(\overline{K}))$). We can choose a simple closed curve representing the preferred longitude $\lambda$ which is invariant under $(f)$; see [12].

Let $\pi : E(K) \to E(\overline{K})$ be the cyclic branched covering branched along $C = C/(f)$.

**Lemma 2.2.** The branched cover $\pi : E(K) \to E(\overline{K})$ can be extended to a branched cover $\pi' : K(\frac{m}{n}) \to \overline{K}(\frac{m}{pn})$.

**Proof of Lemma 2.2.** Let $\pi_* : \pi_1(E(K)) \to \pi_1(E(\overline{K}))$ be the homomorphism induced by $\pi$. Then $\pi_*|_{\pi_1(\partial E(K))} : \pi_1(\partial E(K)) \to \pi_1(\partial E(\overline{K}))$ sends $\mu$ to $\bar{\mu}$, and $\lambda$ to $\bar{\lambda}$. Hence $\pi_*|_{\pi_1(\partial E(K))}(m\mu + n\lambda) = m\bar{\mu} + pn\bar{\lambda} = (m, p)\left\{\frac{m}{(m, p)}\bar{\mu} + \frac{pn}{(m, p)}\bar{\lambda}\right\}$. Then we can extend $\pi : E(K) \to E(\overline{K})$ to $\pi' : K(\frac{m}{n}) = E(K) \cup V \to \overline{K}(\frac{m}{pn}) = E(\overline{K}) \cup \overline{V}$, where $V, \overline{V}$ are filled solid tori. If $(m, p) \geq 2$, then $\pi'$ branches along the core of the filled solid torus $\overline{V}$ as well as $C$. $\square$(Lemma 2.2)

Thus we have a commutative diagram:

$$
\begin{array}{ccc}
E(K) & \xrightarrow{\pi} & E(\overline{K}) \\
\text{Dehn filling} & \downarrow & \text{Dehn filling} \\
K(\frac{m}{n}) & \xrightarrow{\pi'} & \overline{K}(\frac{m}{pn})
\end{array}
$$

Assume that $\frac{m}{pn} \in S_{LO}(\overline{K})$, i.e. $\overline{K}(\frac{m}{pn})$ has left-orderable fundamental group. Let us prove that $K(\frac{m}{n})$ has also left-orderable fundamental group, i.e. $p \times \frac{m}{pn} = \frac{m}{n} \in S_{LO}(K)$.

**Lemma 2.3.** $K(\frac{m}{n})$ is irreducible.

**Proof of Lemma 2.3.** Suppose for a contradiction that $K(\frac{m}{n})$ is reducible. Since $K$ is a nontrivial periodic knot, $K$ is cabled and $\frac{m}{n}$ is the cabling slope [34, 21, 22].
First we assume that $K$ is a torus knot. Then $E(K)$ has a unique Seifert fibration (up to isotopy). Following [35, Theorem 2.2], we choose a Seifert fibration of $E(K)$ which is preserved by $f$. If $C$ is not a fiber, we take a regular fiber $t$ intersecting $C$. Since $f$ fixes a point in $t \cap C$, $f(t) = t$ and $f$ reverses the orientation of $t$. This then implies that $f$ reverses the orientation of $K$, and hence $C$ intersects $K$ in exactly two points, a contradiction. Thus $C$ is a fiber in the $(f)$–invariant Seifert fibration of $E(K)$. Since a regular fiber is knotted in $S^3$, $C$ is one of two exceptional fibers in $E(K)$. Then the quotient $E(\mathcal{K}) = E(K)/\langle f \rangle$ has also a Seifert fibration induced from that of $E(K)$ and thus $\mathcal{K}$ is a torus knot; the surgery slope $\frac{m}{pn}$ on $\partial E(\mathcal{K})$ is the fiber slope. Since $\frac{m}{pn}$ is the fiber (i.e. cabling) slope, $\pi_1(\mathcal{K}(\frac{m}{pn}))$ has a nontrivial torsion, contradicting the left-orderability of $\pi_1(\mathcal{K}(\frac{m}{pn}))$.

Next assume that $K$ is an $(x, y)$–cable in a knotted solid torus $W$, where $y \geq 2$. By the $(f)$–invariant version ([35, Theorem 8.6]) of the torus decomposition theorem [28, 29], we may assume that $f$ leaves a companion solid torus $W$ invariant. First we note that $W \cap C = \emptyset$. For otherwise, $f_{|\partial W}$ has fixed points and hence it is an involution, and $f$ reverses the orientation of an $(f)$–invariant core of $W$. Hence it also reverses the orientation of $K$ (which has winding number $y \geq 2$ in $W$). This then implies that $C$ intersects $K$ in exactly two points, a contradiction. Thus $W \subset S^3 - C$. We denote the quotient $W/\langle f \rangle$ by $\overline{W}$. We may assume that the cable space $W - \text{int}N(K)$ has a Seifert fibration preserved by $f$ [35, Theorem 2.2]. Then $\overline{W} - \text{int}N(\mathcal{K}) = (W - \text{int}N(K))/\langle f \rangle$ has an induced Seifert fibration in which a regular fiber on $\partial N(\mathcal{K})$ represents the surgery slope $\frac{m}{pn}$. This implies that the result of $\frac{m}{pn}$–surgery of $\overline{W}$ along $\mathcal{K}$, and hence $\mathcal{K}(\frac{m}{pn})$, has a nontrivial lens space summand whose fundamental group has order $y \geq 2$. Since $\pi_1(\mathcal{K}(\frac{m}{pn}))$ has a nontrivial torsion, it cannot be left-orderable, contradicting the assumption.

\(\square\) (Lemma 2.3)

The above diagram induces the commutative diagram of fundamental groups below.

\[
\begin{array}{ccc}
\pi_1(E(K)) & \xrightarrow{\pi_*} & \pi_1(E(\mathcal{K})) \\
\downarrow & & \downarrow \\
\pi_1(K(\frac{m}{pn})) & \xrightarrow{\pi'_*} & \pi_1(\mathcal{K}(\frac{m}{pn}))
\end{array}
\]

**Lemma 2.4.** $\pi'_* : \pi_1(K(\frac{m}{pn})) \rightarrow \pi_1(\mathcal{K}(\frac{m}{pn}))$ is surjective.

**Proof of Lemma 2.4.** Choose a point $x \in C = \text{Fix}(f)$ (resp. $\pi(x) \in \overline{C}$) as a base point of $\pi_1(E(K))$ (resp. $\pi_1(E(\mathcal{K}))$). We take obvious meridians $\mu_i$ of $\mathcal{K}$ which are generators of $\pi_1(E(\mathcal{K}), \pi(x))$ (with respect to the Wirtinger presentation of $\pi_1(E(\mathcal{K}), \pi(x))$). Then their lifts $\mu_i \in \pi_1(E(K))$ satisfy $\pi_*(\mu_i) = \mu_i$, and
hence \( \pi_1(E(K)) \to \pi_1(E(K')) \) is an epimorphism. Since vertical homomorphisms are also epimorphisms, \( \pi'_1(K(\frac{m}{n})) \to \pi_1(E(K')) \) is also an epimorphism.

\( \square \) (Lemma 2.4)

By Lemma 2.3 \( K(\frac{m}{n}) \) is irreducible, and by Lemma 2.4 we have an epimorphism from \( \pi_1(K(\frac{m}{n})) \) to the left-orderable group \( \pi_1(E(K')) \). Then it follows from [5, Theorem 1.1(1)] that \( \pi_1(K(\frac{m}{n})) \) is also left-orderable. Thus if \( r = \frac{m}{n} \in S_{LO}(K) \), then \( pr = \frac{m}{n} \in S_{LO}(K) \).

\( \square \) (Theorem 2.1)

3. \( L \)-space surgeries on periodic knots

In [43, 44] Ni proves that if a knot \( K \) in \( S^3 \) has an \( L \)-space surgery, then \( K \) is a fibered knot, i.e. \( E(K) \) has a fibering over the circle. For a periodic knot \( K \), the next theorem gives a necessary condition on the factor knot for \( K \) having an \( L \)-space surgery.

**Theorem 3.1.** Let \( K \) be a periodic knot in \( S^3 \) with axis \( C \), and let \( \overline{K} \) be its factor knot with branch circle \( \overline{C} \). Suppose that \( K \) has an \( L \)-space surgery. Then \( E(K) \) has a fibering over the circle with a fiber surface \( \mathcal{S} \) such that \( |\mathcal{S} \cap \overline{C}| \) equals the algebraic intersection number between \( \mathcal{S} \) and \( \overline{C} \), i.e. the linking number \( lk(\mathcal{S}, \overline{C}) \).

In particular, we have:

**Corollary 3.2.** Let \( K \) be a periodic knot with factor knot \( \overline{K} \). If \( \overline{K} \) is not fibered, then \( S_L(K) = \emptyset \).

**Proof of Theorem 3.1.** Let \( f : S^3 \to S^3 \) be an orientation preserving diffeomorphism of finite order satisfying \( f(K) = K \). Note that \( C = \text{Fix}(f), \overline{K} = K/\langle f \rangle \) and \( \overline{C} = C/\langle f \rangle \). Let \( N(K) \) be an \( \langle f \rangle \)-invariant tubular neighborhood of \( K \).

Assume that \( K \) has an \( L \)-space surgery. Then Ni [43, Corollary 1.3] ([44]) proves that \( E(K) = S^3 - \text{int}N(K) \) has a fibering over the circle. Following Proposition 6.1 in [13], we can isotope the fibering to a fibering preserved by the action of \( \langle f \rangle \) so that the axis \( C \) is transverse to the fibers. Thus \( E(K) \) inherits a fibering over the circle such that all the fibers are transverse to the branch circle \( \overline{C} = C/\langle f \rangle \). Let \( \mathcal{S} \) be a fiber surface of \( E(K) \). Since \( \overline{C} \) intersects each fiber surface of the fibering of \( E(K) \) transversely, \( |\mathcal{S} \cap \overline{C}| \) coincides with the algebraic intersection number between \( \mathcal{S} \) and \( \overline{C} \), i.e. the linking number \( lk(\partial \mathcal{S}, \overline{C}) \), which equals the linking number \( lk(K, \overline{C}) \).

\( \square \) (Theorem 3.1)

As Ni [43, 44] proves, the fiberedness of \( K \) is necessary for \( K \) to have an \( L \)-space surgery. On the other hand, the periodicity of \( K \) itself also puts strong restrictions on 3-manifolds obtained by Dehn surgeries on \( K \). For instance, if a periodic knot \( K \) with period \( p > 2 \) has a finite surgery, which is also an \( L \)-space surgery, then \( K \)
is a torus knot or a cable of a torus knot [38, Proposition 5.6]. So we would like to ask:

**Question 3.3.** Let $K$ be a knot in $S^3$ with cyclic period $p > 2$ other than a torus knot or a cable of a torus knot. Then does $K$ admit an L-space surgery?

## 4. Periodic constructions

Given a periodic knot, taking the quotient by the periodic automorphism, we obtain its factor knot; see Section 2. Reversing this procedure, we have:

**Definition 4.1 (periodic construction).** Let $(\overline{K}, C)$ be a pair of a knot $\overline{K}$ and an unknotted circle $C$ which is disjoint from $\overline{K}$. Then for an integer $p \geq 2$ with $(p, \text{lk}(\overline{K}, C)) = 1$, take the $p$–fold cyclic branched cover of $S^3$ branched along $C$ to obtain a knot $K^p_C$ which is the preimage of $\overline{K}$. We call $K^p_C$ the knot obtained from the pair $(\overline{K}, C)$ by $p$–periodic construction.

Note that $K^p_C$ is a knot with cyclic period $p$ whose factor knot is $\overline{K}$. Hence Theorems 2.1 and 3.1 immediately imply the following result.

**Theorem 4.2.** Let $(\overline{K}, C)$ be a pair as in Definition 4.1. If $\overline{K}$ is a fibered knot, $C$ is chosen so that any fiber surface (i.e. minimal genus Seifert surface) $\mathcal{S}$ satisfies the inequality $|\mathcal{S} \cap C| > \text{lk}(\overline{K}, C)$. Then a knot $K^p_C$ obtained from the pair $(\overline{K}, C)$ by $p$–periodic construction enjoys the following properties:

1. $\mathcal{S}_{\text{LO}}(K^p_C) \supset p\mathcal{S}_{\text{LO}}(\overline{K})$.
2. $\mathcal{S}_I(K^p_C) = \emptyset$.

If $\overline{K}$ is a trivial knot, then $\mathcal{S}_{\text{LO}}(\overline{K}) = \{0\}$ and hence $p\mathcal{S}_{\text{LO}}(\overline{K}) = \{0\}$. So we will apply Theorem 4.2 to nontrivial knots.

**Remark 4.3.** We denote the genus of a knot $k$ in $S^3$ by $g(k)$. For $\overline{K}$ and $K^p_C$, we have $g(K^p_C) \geq pg(\overline{K})$ [42, Theorem 3.2]. In particular, for a nontrivial knot $\overline{K}$, $g(K^p_C) \to \infty$ as $p \to \infty$.

Theorem 4.2 is accompanied by the following theorems.

**Theorem 4.4.** Given a nontrivial knot $\overline{K}$ in $S^3$, we can take an unknotted circle $C$ so that $\overline{K} \cup C$ is a hyperbolic link with arbitrary linking number.

**Proof of Theorem 4.4.** The following argument is based on the proofs of Theorems 2.1 and 2.2 in [1]. Arrange $\overline{K}$ as a closed $n$–braid for some integer $n$. If necessary, stabilizing the braid, we may assume that the braid contains both a positive crossing and a negative crossing (Figure 4.1). Then introduce $(n - 1)$–strands $C_i$
(i = 1, \ldots, n – 1) between the n-strands of the original braid so that the crossings introduced, together with the original crossings, are alternately positive and negative. See Figure 4.1.

![Figure 4.1. Insertion of (n – 1)-strands; n = 2](image)

Then we arrange \( C_i \) as in Figure 4.2 so that the closed braid is a 2-component link consisting of \( K \) and an unknotted circle \( C = C_1 \cup \cdots \cup C_{n-1} \) and \( K \cup C \) is a non-split prime alternating link [37, Theorem 1].

![Figure 4.2. Arrangement of \( C_1, \ldots, C_{n-1}; n = 4 \)](image)

Since our braid contains both a positive crossing and a negative crossing, we can add some negative twists or positive twists as in Figure 4.3 to make \( C \) so that \( \text{lk}(K, C) = l \) for a given integer \( l \).

Note that the resulting link \( K \cup C \) is also a non-split prime alternating link. It follows from [37, Corollary 2] that \( K \cup C \) is either a torus link or a hyperbolic link. Since \( K \) is nontrivial, but \( C \) is trivial, the former cannot occur, and thus \( K \cup C \) is a hyperbolic link.

**Theorem 4.5.**

1. If \( K \cup C \) is a hyperbolic link and \( p > 2 \), then \( K^p_C \) is a hyperbolic knot, and \( K^p_C(r) \) is a hyperbolic 3-manifold for all \( r \in \mathbb{Q} \).

2. Assume that \( p > 2 \) and \( C_i \) (i = 1, 2) is an unknotted circle such that \( \text{lk}(K, C_i) \) and \( p \) are relatively prime, and \( K \cup C_i \) is a hyperbolic link. If \( K^p_{C_1} \) and \( K^p_{C_2} \) are isotopic in \( S^3 \), then \( K \cup C_1 \) and \( K \cup C_2 \) are isotopic.
Proof of Theorem 4.5. (1) Assume for a contradiction that $K_p^C$ is not hyperbolic. Then it is either a torus knot or a satellite knot. Let $f : S^3 \rightarrow S^3$ be the deck transformation of the $p$–fold cyclic branched cover given in Theorem 4.2, which is an orientation preserving diffeomorphism giving the cyclic period $p$ of $K_p^C$. In the following, we take an $(f)$–invariant tubular neighborhood $N(K_p^C)$ and denote $S^3 - \text{int}N(K_p^C)$ by $E(K_p^C)$. The preimage of the branch circle $C$ is an unknotted circle $C = \text{Fix}(f)$, which is contained in the interior of $E(K_p^C)$. Note also that $K_p^C$ is a nontrivial knot. For otherwise, the equivariant loop theorem [36] implies that $K_p^C \cup C$ is the Hopf link and $K \cup C$ is also the Hopf link, contradicting the hyperbolicity of $K \cup C$.

Claim 4.6. $K_p^C$ is not a torus knot.

Proof of Claim 4.6. Assume for a contradiction that $K_p^C$ is a torus knot. Then $E(K_p^C)$ has a unique Seifert fibration up to isotopy. We choose a Seifert fibration of $E(K_p^C)$ which is preserved by $f$ [35, Theorem 2.2]. Then the argument in the proof of Lemma 2.3 shows that $C$ is one of two exceptional fibers in $E(K_p^C)$. Then the quotient $E(K_p^C) - \text{int}N(C) = (E(K_p^C) - \text{int}N(C))/\langle f \rangle$ has also a Seifert fibration. Thus $S^3 - \text{int}N(K \cup C) = E(K) - \text{int}N(C)$ is a Seifert fiber space, contradicting its hyperbolicity. □

Claim 4.7. $K_p^C$ is not a satellite knot.

Proof of Claim 4.7. Suppose for a contradiction that $K_p^C$ is a satellite knot. Then we have an $(f)$–invariant torus decomposition of $E(K_p^C)$ [35, Theorem 8.6]. Let $\Sigma$ be the invariant family of essential tori in $E(K_p^C)$.

Case (i). There is an essential torus $T \in \Sigma$ such that $f(T) = T$. Then $T$ bounds an $(f)$–invariant companion solid torus $W$ containing $K_p^C$. Note that $K_p^C$ is not a core of $W$. We see that $W \cap C = \emptyset$, for otherwise $f|_{\partial W}$ has a fixed point and it is an involution, i.e. $(f|_{\partial W})^2$ is the identity map. By the classical Smith theory [52] $f$
itself is an involution, contradicting the assumption. Thus \( W \) lies in \( S^3 - C \). We may assume that \( W \subset S^3 - \text{int} N(C) \) for a small tubular neighborhood \( N(C) \) of \( C \). Since the core of \( W \) is not a core of \( S^3 - \text{int} N(C) \), \( S^3 - \text{int} N(K^p_W \cup C) \) contains the \((f)\)-invariant essential torus \( T = \partial W \). This then implies that \( S^3 - \text{int} N(K^p_W \cup C) \) contains an essential torus \( \partial W'/(f) \). This contradicts the hyperbolicity of \( S^3 - \text{int} N(K^p_W \cup C) \).

Case (ii). For each \( T \in \Sigma \), \( f(T) \neq T \) (hence, \( f(T) \cap T = \emptyset \)). Let us pick an essential torus \( T \in T \). Note that \( T \) is essential in \( S^3 - \text{int} N(K^p_W \cup C) \). Then the image \( T \subset E(K \cup C) \) of \( T \) by the covering projection is also essential. This contradicts the hyperbolicity of \( S^3 - \text{int} N(K \cup C) \). \( \square \) (Claim 4.7)

It follows that \( K^p_W \) is a hyperbolic knot in \( S^3 \).

Since \( K^p_W \) is a hyperbolic knot with period \( p > 2 \), it follows from [39, Corollary 1.4] that \( K^p_W(r) \) is a hyperbolic 3–manifold for all \( r \in \mathbb{Q} \), or \( p = 3, r = 0 \) and \( g(K^p_W) = 1 \). Since \( g(K^p_W) \geq pq(K) \geq p > 2 \), the latter cannot occur. Hence \( K^p_W(r) \) is a hyperbolic 3–manifold for all \( r \in \mathbb{Q} \) as desired.

(2) In the following, for notational simplicity, we write \( K_i = K^p_{U_i} \).

The assumption, together with (1), implies that \( K_i \) \((i = 1, 2)\) is a hyperbolic knot. Recall that \( K_i \) has an orientation preserving diffeomorphism \( f_i \) such that \( f_i(K_i) = K_i \), \( f_i^0 = id \) and \( \text{Fix}(f_i) = C_i \). Note that \( K = K_i/(f_i) \) and \( C_i/(f_i) \). Suppose that \( K_i \) and \( K_2 \) are isotopic in \( S^3 \). Then we have an orientation preserving diffeomorphism \( \varphi \) of \( S^3 \) such that \( \varphi(K_1) = K_2 \). Note that \( f_2 \varphi^{-1} \circ f_2 \circ \varphi \) is an orientation preserving diffeomorphism of \( S^3 \), which satisfies \( f_2'(K_1) = K_1 \) and gives also a cyclic period \( p \) for \( K_1 \). Let us put \( C_2' = \varphi^{-1}(C_2) \). Then we see that \( \text{Fix}(f_2') = C_2' \). Since \( \varphi \circ f_2 = f_2 \circ \varphi \), \( \varphi \) induces an orientation preserving diffeomorphism \( \varphi \colon S^3 = S^3/f_2 \to S^3 = S^3/f_2 \) sending \( K_1/f_2 \) to \( K_2/f_2 \) and \( C_2/f_2 \) to \( C_2'/(f_2) \).

Now the hyperbolic knot \( K_1 \) has two orientation preserving, periodic diffeomorphisms \( f_1 \) and \( f_2' \) of period \( p > 2 \). Then [3, 2.1 Theorem (a)] shows that the pairwise isotopy classes \([f_1]\) and \([f_2]\) in the symmetry group \( \text{Sym}(S^3) \) have order \( p \). Furthermore, since \( K_1 \) is hyperbolic, \( \text{Sym}(S^3, K_1) \) is isomorphic to a finite cyclic group or a dihedral group [31, Theorems 10.5.3 and 10.6.2(2)]. This implies that subgroups \([f_1]\) and \([f_2]\) of order \( p \) coincide, since \( p > 2 \). Then it follows from [3, 2.1 Theorem (c)] that \([f_1]\) and \([f_2]\) are conjugate by a diffeomorphism \( g \) in \( \text{Diff}(S^3, K_1) \) which is isotopic to the identity. Hence \((f_2')^k = g \circ f_1 \circ g^{-1}\) for some integer \( k \((1 \leq k \leq p - 1)\), which has also period \( p \). Note that \((f_2')^k \) leaves \( K_1 \) invariant and \( K_1/(f_2')^k = K_1/(f_2) \), and that \( \text{Fix}((f_2')^k) = C_2' \) and \( C_2'/(f_2')^k = C_2/(f_2) \). For any \( x \in C_1 = \text{Fix}(f_1) \), we have \((f_2')^k(g(x)) = g(f_1(x)) = g(x)\), thus \( g(x) \in \text{Fix}((f_2')^k) = C_2' \), and hence \( g(C_1) \subset C_2' \). Conversely if \( x' \in C_2' = \text{Fix}((f_2')^k) \), then we see that \( g^{-1}(x') \subset C_1 \) and \( x' \in g(C_1) \), hence \( C_2' \subset g(C_1) \). Thus we
have \( g(C_1) = C_2' \). Therefore we have an orientation preserving diffeomorphism 
\[ g : S^3 = S^3/(f_1) \to S^3 = S^3/(f_2') \] sending \( \overline{K} = K_1/(f_1) \) to \( K_1/(f_2') = K_1/f_2' \) and \( \overline{C_1} = C_1/f_1 \) to \( C_2'/f_2' = C_2/f_2' \).

Now the orientation preserving diffeomorphism \( \varphi \circ \overline{g} \) of \( S^3 \) satisfies \( \varphi \circ \overline{g}(\overline{K}) = K \) and \( \varphi \circ \overline{g}(\overline{C_1}) = C_2' \). Thus \( K \cup C_1 \) and \( K \cup C_2' \) are isotopic. \( \square \)(Theorem 4.5)

5. Examples

In this section, we present two examples illustrating how the periodic construction works according to whether the initial knot \( K \) is fibered or not fibered.

First we apply Theorem 4.2 in the case where \( K \) is not fibered. In such a case we can choose \( C \) arbitrarily with \( lk(K;C) = 0 \) to obtain a knot \( K_p \) having properties (1) and (2) in Theorem 4.2.

Let \( T_n(n \not= 0, \pm 1) \) be the twist knot illustrated in Figure 5.1.

\[ \text{Figure 5.1. The twist knot } T_n \]

Then \( T_n \) is a hyperbolic knot, and since the Alexander polynomial of \( T_n \) is not monic, it is not fibered [6, 8.16 Proposition]. Suppose that \( n > 1 \). Then it follows from [56, 20] that \( \pi_1(T_n(r)) \) is left-orderable for \( r \in (-4n, 4) \). Furthermore, it is known by [54] that \( \pi_1(T_n(4)) \) is left-orderable. Hence \( S_{LO}(T_n) \supset (\text{-}8p, 4p] \cap \mathbb{Q} \).

**Example 5.1.** Let us take a 2–component link \( T_2 \cup C \) as in Figure 5.2; \( lk(T_2, C) = 1 \). Let \( p \) be any integer with \( p > 2 \) and \( K_{2,T}^p \) a knot obtained from \( (T_2, C) \) by \( p \)-periodic construction. Then \( K_{2,T}^p \) enjoys the following properties:

1. \( K_{2,T}^p \) is a hyperbolic knot in \( S^3 \).
2. \( K_{2,T}^p(r) \) is a hyperbolic 3–manifold for all \( r \in \mathbb{Q} \).
3. \( S_{LO}(K_{2,T}^p) \supset (\text{-}8p, 4p] \cap \mathbb{Q} \).
4. \( S_{L}(K_{2,T}^p) = \emptyset \).

**Proof.** Assertions (1) and (2) follow from Theorem 4.5(1) once we show that \( T_2 \cup C \) is a hyperbolic link. Since \( T_2 \cup C \) is a non-split prime alternating link [37, Theorem 1], it is either a torus link or a hyperbolic link [37, Corollary 2]. The former cannot happen, because \( T_2 \) is nontrivial, but \( C \) is trivial. Hence \( T_2 \cup C \) is a hyperbolic link.
as desired. Since $T_2$ is not fibered and $\pi_1(T_2(r))$ is left-orderable for $r \in (-8, 4]$, assertions (3) and (4) follow from Theorem 4.2.

Next we apply Theorem 4.2 in the case where $K$ is a fibered knot. In the next example we take a trefoil knot $T_{-3,2}$ as $K$.

**Example 5.2.** Let us take the 2–component link $T_{-3,2} \cup \overline{C}$ shown in Figure 5.3; $\text{lk}(T_{-3,2}, \overline{C}) = 1$. Let $p$ be any integer with $p > 2$ and $K_{-3,2, \overline{C}}^p$ a knot obtained from $(T_{-3,2}, \overline{C})$ by $p$–periodic construction. Then $K_{-3,2, \overline{C}}^p$ enjoys the following properties:

1. $K_{-3,2, \overline{C}}^p$ is a hyperbolic knot in $S^3$.
2. $K_{-3,2, \overline{C}}^p(r)$ is a hyperbolic 3–manifold for all $r \in \mathbb{Q}$.
3. $S_{\text{LO}}(K_{-3,2, \overline{C}}^p) \supset (-p, \infty) \cap \mathbb{Q}$.
4. $S_{\text{L}}(K_{-3,2, \overline{C}}^p) = \emptyset$.

**Figure 5.3.** The trefoil knot $T_{-3,2}$ and the unknotted circle $\overline{C}$

**Proof of Example 5.2.** Recall that $S_{\text{LO}}(T_{-3,2}) = (-1, \infty) \cap \mathbb{Q}$; see Remark 1.2(2) and Example 1.6.

Since as illustrated in Figure 5.3(i) $T_{-3,2} \cup \overline{C}$ is a non-split prime alternating link [37, Theorem 1], it is either a torus link or a hyperbolic link [37, Corollary 2].
If we have the former case, then $T_{-3,2}$ is isotopic to $C$ which is a trivial knot, a contradiction. Hence $T_{-3,2} \cup C$ is a hyperbolic link. Then (1) and (2) follow from Theorem 4.5(1).

Let us prove (3) and (4) using Theorem 4.2. Since $T_{-3,2}$ is fibered, we need to check the condition of Theorem 4.2: for any fiber surface $S$ of $E(T_{-3,2})$, $|S \cap C|$ is strictly bigger than the algebraic intersection number between $S$ and $C$, i.e. $\text{lk}(T_{-3,2}, C)$.

In Figure 5.3(ii), we give a minimal genus Seifert surface $F$ of $T_{-3,2}$, which is a once-punctured torus with $\partial F = T_{-3,2}$. Put $\overline{S} = F \cap E(T_{-3,2})$. Then by [13, Lemma 5.1] $\overline{S}$ is a fiber surface of $E(T_{-3,2})$. We see that $|\overline{S} \cap C| = 5$ and the algebraic intersection number between $\overline{S}$ and $C$ is one. Assume for a contradiction that we have another fiber surface $\overline{S}'$ of $E(T_{-3,2})$ such that $|\overline{S}' \cap C| < |\overline{S} \cap C|$. Since $\overline{S}$ and $\overline{S}'$ are fiber surfaces of $E(T_{-3,2})$, they are isotopic; see [13, Lemma 5.1], [55]. This then implies that we can isotope $C$ to $\overline{C}'$ in $E(T_{-3,2})$ so that $|\overline{S}' \cap C| < |\overline{S} \cap C|$.

**Claim 5.3.** There exists a smooth map $\varphi$ from a semi-disk $D$ into $E(T_{-3,2})$ such that $\varphi^{-1}(C)$ is an arc $c \subset \partial D$ and $\varphi^{-1}(\overline{S})$ is the arc $\alpha = \partial D - c$.

**Proof of Claim 5.3.** Let $\Phi : S^1 \times [0,1] \to E(T_{-3,2})$ be a smooth map giving an isotopy between $\overline{C} (= \Phi(S^1 \times \{0\}))$ to $\overline{C}' (= \Phi(S^1 \times \{1\}))$. We may assume $\Phi$ is transverse to $\overline{S}$. Furthermore, the essentiality of $\overline{S}$ in $E(T_{-3,2})$ enables us to modify $\Phi$ to eliminate the circle components as usual. Since $|\overline{S} \cap C| < |\overline{S} \cap C| = 5$ and the algebraic intersection number between $\overline{S}$ and $C$, we have $|\overline{S} \cap C| = 1$ or 3. Thus $\Phi^{-1}(\overline{S})$ consists of three properly embedded arcs $\alpha$, $\alpha'$, and $\beta$, where $\partial \alpha \subset S^1 \times \{0\}$, $\partial \alpha' \subset S^1 \times \{0\}$, and $\beta$ connects $S^1 \times \{0\}$ and $S^1 \times \{1\}$ (Figure 5.4(ii), (iii)), consists of four properly embedded arcs $\alpha$, $\alpha'$, and $\beta'$, where $\partial \alpha \subset S^1 \times \{0\}$, and each of $\beta, \beta', \beta''$ connects $S^1 \times \{0\}$ and $S^1 \times \{1\}$ (Figure 5.4(iii)), or consists of four properly embedded arcs $\alpha$, $\alpha'$, $\beta$, and $\gamma$, where $\partial \alpha \subset S^1 \times \{0\}$, $\partial \alpha' \subset S^1 \times \{0\}$, $\beta$ connects $S^1 \times \{0\}$ and $S^1 \times \{1\}$, and $\partial \gamma \subset S^1 \times \{1\}$ (Figure 5.4(iv), (v)). In either case there is a semi-disk $D$ cobounded by $\alpha$ and an arc $c \subset S^1 \times \{0\}$.

Putting $\varphi = \Phi|_D : D \to E(T_{-3,2})$, we obtain a desired smooth map. $\square$(Claim 5.3)

Cut open $E(T_{-3,2})$ along $\overline{S}$ to obtain a product 3–manifold $\overline{S} \times [0,1]$. The circle $C$ is cut into five arcs $c_1, c_2, c_3, c_4$ and $c_5$ as in Figure 5.3(ii). Note that $\partial c_1 \subset \overline{S} \times \{0\}$, $\partial c_3 \subset \overline{S} \times \{1\}$, and each of $c_2, c_4, c_5$ connects $\overline{S} \times \{0\}$ and $\overline{S} \times \{1\}$. Moreover, we see that $c_1$ and $c_3$ are linking once relative their boundaries.

On the other hand, since $c$ is either $c_1$ or $c_3$, Claim 5.3 shows that $c_1$ and $c_3$ are unlinked relative their boundaries. This contradiction shows that for any fiber surface $\overline{S}$, $|\overline{S} \cap C| = 5$ and $|\overline{S} \cap C| > \text{lk}(T_{-3,2}, C)$. 


Since $\pi_1(T_{-3,2}(r))$ is left-orderable if $r \in (-1, \infty)$, the conclusions (3) and (4) follow from Theorem 4.2. This completes the proof of Example 5.2. □ (Example 5.2)

6. SURGERIES ON ALTERNATING KNOTS

Theorem 1.5 in [47], together with [48, Proposition 9.6] ([43, Proof of Corollary 1.3], [26, Claim 2]), shows that for an alternating knot $K$ which is not a $(p, 2)$–torus knot, $K(r)$ is not an $L$–space for all $r \in \mathbb{Q}$.

We say that an alternating knot is positive (resp. negative) if it has a reduced alternating diagram such that each of the crossings is positive (resp. negative). An alternating knot is special if it is either positive or negative.

In [4] Boyer, Gordon and Watson prove:

**Proposition 6.1** ([4]). Let $K$ be a prime alternating knot in $S^3$.

1. If $K$ is not a special alternating knot, then $\pi_1(K(\frac{1}{n}))$ is left-orderable for all non-zero integers $n$.

2. If $K$ is a positive (resp. negative) alternating knot, then $\pi_1(K(\frac{1}{n}))$ is left-orderable for all positive (resp. negative) integers $n$.

Let $\overline{K}$ be an alternating knot. For convenience, we position $\overline{K} \subset \mathbb{R}^3 = S^3 - \{\infty\}$ so that $\overline{K}$ lies in the $xy$–plane except near crossings of $\overline{K}$, where $\overline{K}$ lies on a “bubble” as in [37]. Then we say an unknotted circle $\overline{C} \subset S^3 - \overline{K}$ is perpendicular if it passes $\infty$ and intersects the $xy$–plane exactly once. Note that $\overline{C} \cap \mathbb{R}^3$ is perpendicular to the $xy$–plane. See Figure 6.1, in which the dot indicates a perpendicular circle $\overline{C}$. 
Figure 6.1. An alternating knot $\mathcal{K}$ and a perpendicular circle $\mathcal{C}$

**Proposition 6.2.** Let $\mathcal{K}$ be a prime alternating knot and $\mathcal{C}$ a perpendicular circle. Let $p$ be an integer such that $p \geq 2$ and $(p, \text{lk}(\mathcal{K}, \mathcal{C})) = 1$, and let $K_{\mathcal{C}}^p$ be the knot obtained from $(\mathcal{K}, \mathcal{C})$ by $p$–periodic construction. Then we have:

1. $K_{\mathcal{C}}^p$ is an alternating knot.
2. If $\mathcal{K}$ is not a special alternating knot, then $\pi_1(K_{\mathcal{C}}^p(\frac{r}{n}))$ is left-orderable for all non-zero integers $n$.
3. If $\mathcal{K}$ is a positive (resp. negative) alternating knot, then $\pi_1(K_{\mathcal{C}}^p(\frac{r}{n}))$ is left-orderable for all positive (resp. negative) integers $n$.

**Proof of Proposition 6.2.** The first assertion follows immediately from diagramatic consideration. The conclusions (2) and (3) follow from Proposition 6.1 and Theorem 2.1.

**Remark 6.3.** In Proposition 6.2, if $\mathcal{K}$ is not a $(p, 2)$–torus knot, then $K_{\mathcal{C}}^p$ is not a $(p', 2)$–torus knot. For otherwise, the argument in the proof of Lemma 2.3 implies that $\mathcal{K}$ is a torus knot. Since it is alternating, it is a $(p, 2)$–torus knot for some odd integer $p$ [41, Theorem 3.2], a contradiction. Therefore, as mentioned in the beginning of this section, $K_{\mathcal{C}}^p(r)$ is not an $L$–space for all $r \in \mathbb{Q}$.

Applying Proposition 6.2 and Remark 6.3, we have:

**Example 6.4.** Take an alternating knot $\mathcal{K}$ and a perpendicular circle $\mathcal{C}$ as illustrated in Figure 6.1; $\text{lk}(\mathcal{K}, \mathcal{C}) = 1$. Note that $\mathcal{K}$ is not a special alternating knot. Hence for any integer $p \geq 2$, $K_{\mathcal{C}}^p$ is an alternating knot, $K_{\mathcal{C}}^p(\frac{r}{n})$ is not an $L$–space for all $r \in \mathbb{Q}$, and $\pi_1(K_{\mathcal{C}}^p(\frac{r}{n}))$ is left-orderable for all non-zero integers $n$.

7. Knots with $S_{LO}(K) = \mathbb{Q}$ and $S_{L}(K) = \emptyset$

The goal of this section is to prove Theorems 1.8 and 1.9. We start with Proposition 7.1 below, which was shown by Clay and Watson [10, Proposition 4.1].
Let $k$ be a knot in $S^3$, which is contained in a standardly embedded solid torus $V \subset S^3$. Assume that $k$ is not contained in a 3-ball in $V$. We call $k$ a \textit{pattern knot} in $S^3$ and the pair $(V, k)$ a \textit{pattern}. Let $f$ be an orientation preserving embedding from $V$ into $S^3$ which sends a preferred longitude of $V$ to that of $f(V) \subset S^3$. Then we obtain a knot $K = f(k)$ in $S^3$, which is called a \textit{satellite knot} with a pattern knot $k$ and a companion knot $K' = f(c)$, where $c$ is a core of $V$.

**Proposition 7.1** ([10]). Let $K$ be a satellite knot with a pattern knot $k$. If $K(r)$ is irreducible and $r \in S_{LO}(k)$, then $r \in S_{LO}(K)$.

7.1. Composite knots $K$ with $S_{LO}(K) = \mathbb{Q}$ and $S_{L}(K) = \emptyset$. In this subsection we prove that the connected sum of two torus knots $T_{-p,q}$ and $T_{r,s}$ where $p > q \geq 2$ and $r > s \geq 2$, satisfies $S_{LO}(T_{-p,q} \# T_{r,s}) = \mathbb{Q}$ and $S_{L}(T_{-p,q} \# T_{r,s}) = \emptyset$ (Proposition 7.3). Thus $T_{-p,q} \# T_{r,s}$ satisfies Conjecture 1.3.

Proposition 7.1 and Theorem 2.1 immediately imply:

**Proposition 7.2.** Let $K$ and $K'$ be nontrivial knots. Then we have:

1. $S_{LO}(K \# K') \supset S_{LO}(K) \cup S_{LO}(K')$.
2. $S_{LO}(pK) \supset pS_{LO}(K)$, where $pK$ denotes the connected sum of $p$ copies of $K$.

**Proof of Proposition 7.2.** (1) Following [17, Lemma 7.1], we see that $(K \# K')(r)$ is irreducible for all $r \in \mathbb{Q}$.

Let us regard $K \# K'$ as a satellite knots with a pattern knot $K$ and a companion knot $K'$. Then Proposition 7.1 shows that $S_{LO}(K \# K') \supset S_{LO}(K)$. Exchanging the roles of $K$ and $K'$, we have $S_{LO}(K \# K') \supset S_{LO}(K')$ as well. Thus $S_{LO}(K \# K') \supset S_{LO}(K) \cup S_{LO}(K')$.

(2) Since $pK$ is a knot with cyclic period $p$ whose factor knot is $K$, the result follows from Theorem 2.1. \qed

As a step toward proofs of Theorems 1.8 and 1.9, we prove:

**Proposition 7.3.** For torus knots $T_{-p,q}$ and $T_{r,s}$, where $p > q \geq 2$ and $r > s \geq 2$, we have $S_{LO}(T_{-p,q} \# T_{r,s}) = \mathbb{Q}$ and $S_{L}(T_{-p,q} \# T_{r,s}) = \emptyset$.

**Proof of Proposition 7.3.** Recall that $S_{LO}(T_{-p,q}) = (-pq + p + q, \infty) \cap \mathbb{Q}$ and $S_{LO}(T_{r,s}) = (-\infty, rs - r - s) \cap \mathbb{Q}$. Note that $-pq + p + q < 0 < rs - r - s$. Now apply Proposition 7.2 to $T_{-p,q} \# T_{r,s}$ to conclude that $S_{LO}(T_{-p,q} \# T_{r,s}) \supset ((-pq + p + q, \infty) \cup (-\infty, rs - r - s)) \cap \mathbb{Q} = \mathbb{Q}$. Hence $S_{LO}(T_{-p,q} \# T_{r,s}) = \mathbb{Q}$.

Next we show that $T_{-p,q} \# T_{r,s}$ has no $L$-space surgeries.

**Claim 7.4.** The coefficient of $t$ in the Alexander polynomial of $T_{-p,q} \# T_{r,s}$ is $-2$. 


**Proof of Claim 7.4.** Recall that $T_{p,q}$ has the Alexander polynomial $\Delta_{T_{p,q}}(t) = \frac{(t^{p+1} - 1)(t^p - 1)}{(t^q - 1)(t^q - 1)}$, and $T_{r,s}$ has the Alexander polynomial $\Delta_{T_{r,s}}(t) = \frac{(t^{r+1} - 1)(t^r - 1)}{(t^s - 1)(t^s - 1)}$. Since $\Delta_{T_{p,q}}(t) \Delta_{T_{r,s}}(t) = \Delta_{T_{p,q}}(t) \Delta_{T_{r,s}}(t)$, $\Delta_{T_{p,q}}(0) = \Delta_{T_{r,s}}(0) = 1$ and a simple computation shows that $\Delta_{T_{p,q}}(0) = \Delta_{T_{r,s}}(0) = -1$. This then implies that the coefficient of $t$ in the Alexander polynomial of $T_{p,q} \natural T_{r,s}$ is $-2$. 

Apply [47, Corollary 1.3], together with [48, Proposition 9.6] ([43, Proof of Corollary 1.3], [26, Claim 2]), to conclude that $T_{p,q} \natural T_{r,s}$ has no $L$–space surgeries.

**Proposition 7.3**

Let us consider the connected sum $T_{p,q} \natural T_{r,s}$ instead of $T_{p,q} \natural T_{r,s}$, where $p > q \geq 2$ and $r > s \geq 2$. The argument in the proof of Claim 7.4 shows that $\mathcal{S}_L(T_{p,q} \natural T_{r,s}) = \emptyset$. On the other hand, putting $m_0 = \max\{pq - p - q, rs - r - s\}$, Example 1.6 and Proposition 7.2 merely imply $\mathcal{S}_{LO}(T_{p,q} \natural T_{r,s}) \supset (-\infty, m_0)$. So we would like to ask:

**Question 7.5.** Does $\mathcal{S}_{LO}(T_{p,q} \natural T_{r,s}) = \mathbb{Q}$ hold for integers $p > q \geq 2$ and $r > s \geq 2$?

### 7.2. Proof of Theorem 1.8.

Let us consider $k = T_{-3,2} \natural T_{3,2}$ and take an unknotted circle $C$ as in Figure 7.1. Following Proposition 7.3, $\mathcal{S}_{LO}(k) = \mathbb{Q}$.

![Figure 7.1. $k \cup C$](image)

Note that the link $k \cup C$ is an alternating link. Since $k \cup C$ is a non-split prime alternating link [37, Theorem 1], it is either a torus link or a hyperbolic link [37, Corollary 2]. The former is not the case, because $k$ is nontrivial, but $C$ is trivial. Thus $k \cup C$ is hyperbolic, hence letting $V = S^3 - \text{int}N(C)$, $k$ is a hyperbolic knot in $V$. Apply the satellite construction with the pattern $(V, k)$ and the companion knot $K'$ to obtain a satellite knot $K$ with a pattern knot $k = T_{-3,2} \natural T_{3,2}$. Since $k$ is hyperbolic in $V$, the satellite knot $K$ is prime, and the 3–manifold obtained from
V by r–surgery on k is again hyperbolic for all but finitely many \( r \in \mathbb{Q} \). This then implies that \( K(r) \) is a toroidal 3–manifold with a hyperbolic piece, in particular, \( K(r) \) is not a graph manifold, for all but finitely many \( r \in \mathbb{Q} \). This establishes (1).

If \( K(r) \) were reducible for some \( r \in \mathbb{Q} \), then \( K \) is cabled [51, 4.5 Corollary]. However, this is impossible, because \( V - k \) is hyperbolic. Hence \( K(r) \) is irreducible for any \( r \in \mathbb{Q} \). Now Proposition 7.1 shows that \( S_{LO}(K) \supseteq S_{LO}(k) = \mathbb{Q} \).

Let us see that \( S_{L}(K) = 0 \). Since \( \text{lk}(k, C) = 0 \), i.e. the winding number of \( K \) in \( V \) is zero, \((V, k)\) is not fibered, and hence neither is the satellite knot \( K \); see [25, Theorem 1]. Hence [43, Corollary 1.3] shows that \( S_{L}(K) = 0 \).

Finally we show that there are infinitely many satellite knots \( K \) with a companion knot \( K' \) and enjoy the required properties in Theorem 1.8. For instance, let us take \( k_p = T_{-3,2} \# T_{p,2} \) (\( p \geq 3 \)). As shown in Proposition 7.3, \( S_{LO}(k_p) = \mathbb{Q} \) for all \( p \geq 3 \). It follows from Theorem 4.4, there is an unknotted circle \( C_p \), so that \( k_p \cup C_p \) is hyperbolic and \( \text{lk}(k_p, C_p) = 0 \). Each \( C_p \) gives a pattern \((V, k_{p})\). Let \( K_p \) be a satellite knot with a companion knot \( K' \) and pattern \((V, k_{p})\). Then the same argument as above shows that \( K_p \) satisfies the properties of Theorem 1.8. If \( p \neq p' \geq 3 \), then \( k_p \cup C_p \) are not isotopic to \( k_{p'} \cup C_{p'} \). Hence there is no orientation preserving diffeomorphism of \( V \) which leaves the preferred longitude of \( V \) invariant and maps \( k_p \) to \( k_{p'} \). Thus we see that the resulting satellite knots \( K_p \) and \( K_{p'} \) are never isotopic.

\( \Box \) (Theorem 1.8)

7.3. Proof of Theorems 1.9. As in the proof of Theorem 1.8, we consider the connected sum \( T_{-3,2} \# T_{3,2} \), which has the property: \( S_{LO}(T_{-3,2} \# T_{3,2}) = \mathbb{Q} \) (Proposition 7.3).

Although we can apply the periodic construction and Theorem 4.2 to the fibered knot \( T_{-3,2} \# T_{3,2} \), for ease of handling, we take the connected sum \((T_{-3,2} \# T_{3,2}) \# T_2\), where \( T_2 \) is the twist knot shown in Figure 5.1. The Alexander polynomial of \((T_{-3,2} \# T_{3,2}) \# T_2\) is \((t^2 - t + 1)^2(2t^2 - 5t + 2)\), which is not monic, and hence \((T_{-3,2} \# T_{3,2}) \# T_2\) is not fibered. Proposition 7.2 shows that \( S_{LO}((T_{-3,2} \# T_{3,2}) \# T_2) \supset S_{LO}(T_{-3,2} \# T_{3,2}) \supset S_{LO}(T_{-3,2} \# T_{3,2}) = \mathbb{Q} \), and hence \( S_{LO}((T_{-3,2} \# T_{3,2}) \# T_2) = \mathbb{Q} \).

Let us put \( \overline{K} = T_{-3,2} \# T_{3,2} \# T_2 \) and take an unknotted circle \( \overline{C} \) as in Figure 7.2; \( \text{lk}(\overline{K}, \overline{C}) = 1 \).

Since \( \overline{K} \cup \overline{C} \) is a non-split prime alternating link [37, Theorem 1], it is either a torus link or a hyperbolic link [37, Corollary 2]. The former cannot happen, because \( \overline{K} \) is nontrivial, but \( \overline{C} \) is trivial. Hence \( \overline{K} \cup \overline{C} \) is a hyperbolic link. Let \( p > 2 \) be any integer, and apply the \( p \)–periodic construction to the pair \((\overline{K}, \overline{C})\) to obtain a knot \( \overline{K}_p \). It follows from Theorem 4.2 and Theorem 4.5(1) that \( \overline{K}_p \) is a hyperbolic knot and enjoys the properties (1), (2) and (3) in Theorem 1.9. By changing \( p \), we obtain infinitely many such knots. For instance, see Remark 4.3. \( \Box \) (Theorem 1.9)
Remark 7.6.  
(1) By Theorem 4.4 there are infinitely many unknotted circles for \( K = T_{3,2} \# T_{3,2} \# T_2 \), and for each unknotted circle \( C \) we obtain infinitely many hyperbolic knots \( K^p C \), where \( p \) and \( \text{lk}(K, C) \) are relatively prime. See also Theorem 4.5(2).

(2) Recall that any knot \( K \) obtained by the periodic construction (Section 4), for instance a knot obtained in the proof of Theorem 1.9, is not fibered and every nontrivial surgery on \( K \) is a left-orderable, non-\( L \)-space surgery. So we can apply Theorem 4.2 again to the knot \( K \) and an arbitrarily chosen unknotted circle to obtain yet further infinitely many non-fibered knots \( K' \) each of which has the (same) factor knot \( K \). Then \( r \)-surgery on \( K' \) is also a left-orderable, non-\( L \)-space surgery for all \( r \in \mathbb{Q} \). We can apply this procedure repeatedly arbitrarily many times.

(3) Let \( K \) be the knot 10\text{99} in Rolfsen’s knot table [50]. Recently Clay [7] used an epimorphism from \( E(K) \) to \( E(T_{3,2}) \) which preserves the peripheral subgroup [32] to show that every nontrivial surgery on \( K \) is left-orderable surgery. Since \( K \) has no cyclic period [31, Appendix F], this example cannot be explained by the periodic construction.

8. Shapes of \( S_{\text{LO}}(K) \) – Questions and Conjectures

As we mentioned in Remark 1.2(1), \( 0 \in S_{\text{LO}}(K) \) for any knot \( K \). If \( K \) is the trivial knot then \( S_{\text{LO}}(K) = \{0\} \), which has the smallest size. On the other hand, Theorems 1.8, 1.9 and Proposition 7.3 demonstrate that there are infinitely many knots \( K \) with \( S_{\text{LO}}(K) = \mathbb{Q} \), which has largest size.

It seems interesting to determine the shape of \( S_{\text{LO}}(K) \) when it is neither \( \{0\} \) nor \( \mathbb{Q} \).

Question 8.1. If \( K \) is a nontrivial knot in \( S^3 \), then does \( S_{\text{LO}}(K) \) contain \((-1, 1) \cap \mathbb{Q}\)?

Recently Li and Roberts [33, Corollary 1.2] prove that for any hyperbolic knot \( K \), there exists a constant \( N_K \) such that \( \{ \frac{1}{n} \mid |n| > N_K \} \subset S_{\text{LO}}(K) \).
More strongly, we would like to ask:

**Question 8.2.** If $K$ is a nontrivial knot in $S^3$, then does $S_{LO}(K)$ contain $(-\infty, 1) \cap \mathbb{Q}$ or $(-1, \infty) \cap \mathbb{Q}$?

For the simplest nontrivial knot $T_{3,2}$ (resp. $T_{-3,2}$), we have $S_{LO}(T_{3,2}) = (-\infty, 1) \cap \mathbb{Q}$ (resp. $S_{LO}(T_{-3,2}) = (-1, \infty) \cap \mathbb{Q}$); see Remark 1.2(2) and Example 1.6.

**Question 8.3.** If $S_{LO}(K) = (-\infty, 1) \cap \mathbb{Q}$ or $S_{LO}(K) = (-1, \infty) \cap \mathbb{Q}$, then is $K$ a trefoil knot $T_{3,2}$ or $T_{-3,2}$, respectively?

**Question 8.4.** Let $K$ be a nontrivial knot in $S^3$. Then does $S_{LO}(K)$ have a maximum or minimum?

Conjecture 1.3 says that $S_L(K)$ and $S_{LO}(K)$ are complementary to each other in $\mathbb{Q}$ if $K$ is not a cable of a nontrivial knot. So let us look at the shape of $S_L(K)$, which is described by Proposition 9.6 in [48] ([23, Lemma 2.13]).

**Theorem 8.5** ([48, 23]). Suppose that $K$ is a nontrivial knot and $S_L(K) \neq \emptyset$. Then $S_L(K) = [2g(K) - 1, \infty) \cap \mathbb{Q}$ or $S_L(K) = (-\infty, -2g(K) + 1] \cap \mathbb{Q}$.

Theorem 8.5 makes us expect the following explicit form of $S_{LO}(K)$.

**Conjecture 8.6.** Let $K$ be a nontrivial knot in $S^3$ which is not a cable of a nontrivial knot. Then $S_{LO}(K)$ coincides with one of $\mathbb{Q}$, $(-\infty, 2g(K) - 1) \cap \mathbb{Q}$ or $(-2g(K) + 1, \infty) \cap \mathbb{Q}$.

Finally we give a comment on Question 8.3 in case of $S_{LO}(K) = (-\infty, 1) \cap \mathbb{Q}$; the other case follows by taking the mirror image. By the assumption $1 \notin S_{LO}(K)$. If Conjecture 1.1 is true, then $1 \in S_L(K)$ or $K(1)$ is reducible. The latter possibility is eliminated by [18, Corollary 3.1], and hence $K(1)$ is an $L$-space. Then Proposition 8.7 [24, Proposition 6] below shows that $K$ is a trefoil knot $T_{3,2}$.

**Proposition 8.7** ([24]). Suppose $K$ is a nontrivial knot and $K(\frac{1}{n})$ is an $L$-space. Then $n = 1$ (resp. $-1$) and $K$ is a trefoil knot $T_{3,2}$ (resp. $T_{-3,2}$).

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