Introduction to Knot Framing Functions

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結び目の数学 VI
Before starting: convention

\[ K : S^1 \hookrightarrow S^3 : \text{Smooth, oriented knot in } S^3 \]

**Convention**

The word “knot” may mean:

1. A particular smooth embedding of \( S^1 \) into \( S^3 \) (“knot” regarded as a smooth map)
2. An image of a particular smooth embedding of \( S^1 \) (Usual meaning of knots, and “knot” as a submanifold)
3. An isotopy (diffeotopy) class of specified embedding of circle. (Usual meaning of isotopy class of knots)
Compressing disc

**Definition**

For a knot $K : S^1 \hookrightarrow S^3$, a *compression disc* of $K$ is a smooth map $D : D^2 \rightarrow S^3$ which satisfies:

1. $D|_{\partial D^2} = K$.
2. $D|_{\text{Int} D^2}$ transverse $K$.

Remark ▶ $K$ is null-homotopic ($\pi_1(S^3) = 1$), so a compression disc always exists. (D is far from embedding)

▶ A compression disc $D$ may not be an immersion, but one may assume that singularity of $D$ is either branch point of Whitney's umbrellas (a generic map (general position) theory).

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- A compression disc $D$ may not be an immersion, but one may assume that singularity of $D$ is either branch point of Whitney’s umbrellas (a generic map (general position) theory).
Compressing disc: Example

A typical example of compressing disc is a clasp disc: A disc $D$ bounded by a knot $K$ having only the clasp singularities.
Framing function

Definition (Greene-Wiest, ‘98)

For a knot $K$, the **framing function** is a map

$$n_K : \mathbb{Z} \to \mathbb{N}$$

defined by

$$n_K = \min \{ \# K \cap D \mid D \text{ is a compressing disc of } K \text{ with } i(K, D) = k \}$$

Here,

\[
\begin{align*}
  i(K, D) & \text{ denotes the algebraic intersection number of } K \text{ and } D \\
  \# K \cap D & \text{ denotes the geometric intersection number of } K \text{ and } D.
\end{align*}
\]
Basic properties and related invariants

Proposition (Greene-Wiest)

For a $K \subset S^3$,

1. $n_K(k) \geq |k|$.
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3. $n_K(0)$ is attained by an immersed compressing disc.
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3. $n_K(0)$ is attained by an immersed compressing disc.

4. $\nu(K) = \lim_{k \to \infty} \frac{n_K(k) - n_K(-k)}{2}$ is well-defined and $\nu(K) \in \mathbb{Z}$. $\nu(K)$ is called a natural framing of $K$. 

Complexity $L(K)$ defined as $\min_{k \in \mathbb{Z}} n_K(k)$ is called a complexity of knot $K$. $L(K)$ always takes an even integers.
Basic properties and related invariants

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Complexity

$L(K) \overset{\text{Def}}{=} \min_{k \in \mathbb{Z}} n_K(k)$ is called a complexity of knot $K$. $L(K)$ always takes an even integers.
Example 1: Figure eight knot

How can we determine the framing function?

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$K$: Figure eight knot

1. $n_K(2) = n_K(-2) = 2.$
   
   Rem: $(n_K(k) \geq |k|)$ so it is sufficient to find a compressing disc with only positive/negative intersections.
2. Basic properties determine $n_K(k)$ except $k = 0$. 
   $(n_K(0) = 4$ or $2$. To determine $n_K(0)$, we need more deep argument.)

3. $\nu(K) = 0$ and $L(K) = 2$. 
More generally:

**Proposition**

1. \( n_K(k) = n_{mK}(-k) \), where \( mK \) is the mirror image of \( K \).
2. In particular, \( \nu(K) = -\nu(mK) \), hence if \( K \) is amphicheiral, then \( \nu(K) = 0 \).
   
   (c.f. For amphicheiral knot \( K \), \( \sigma(K) = 0 \).)
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- It is hard to determine or even to have a non-trivial estimation for \( n_K \) or \( \mathcal{L}(K) \)
- It is often easier to handle \( \nu(K) \).
Example 2: Torus knot

Theorem (Greene-Wiest)

\[ K = K(p, q): (p, q)\text{-torus knot} \]

\[ n_{K(p, q)}(k) = (p - 1)(q - 1) + |k + (p - 1)(q - 1)| \]
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Remark

Recall that by Milnor's conjecture (first proven by Kronheimer-Mrowka)

\[ u(K) = g_4(K) = g(K) = \frac{(p - 1)(q - 1)}{2} \]

The proof uses geometric aspects of Cayley graph of the torus knot group and algebraic interpretation of \( n_K \).
Algebraic interpretation

Fix a meridian $\mu \in \pi_1(S^3 - K)$.

$M = \{g\mu \pm g^{-1} | g \in \pi_1(S^3 - K)\}$ : Set of meridinal elements

$l_k = \text{longitude of } K \text{ with } lk(K, l_k) = k \ (\in \pi_1(S^3 - K))$
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**Theorem (Greene-Wiest '98)**

\[ n_K(k) = l_{\mathcal{M}}(l_k). \]

Here \( l_{\mathcal{M}} \) is the length function of \( \pi_1(S^3 - K) \) with respect to the generating set \( \mathcal{M} \).
Algebraic interpretation

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**Theorem (Greene-Wiest ’98)**

$$n_K(k) = l_\mathcal{M}(l_k).$$

Here $l_\mathcal{M}$ is the length function of $\pi_1(S^3 - K)$ with respect to the generating set $\mathcal{M}$.

cf. **Theorem (Calegari-Gabai)**

$$g(K) = l_{[G,G]}(l_0)$$

Here $G = \pi_1(S^3 - K)$ and $l_{[G,G]}$ is the commutator length: the length function of $G$ with respect to the commutators $[G, G]$. 
Idea of proof: Compression disc provides a factorization of a longitude as a product of meridinal elements and vice versa.
Motivating question

Question

Determine or estimate $n_K(k)$, $\nu(K)$, or $\mathcal{L}(k)$.
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2. $\nu(K)$ often coincides with the signature $\sigma(K)$. However, there are knots $K$ with $\nu(K) \neq \sigma(K)$.
   (In particular, there is a knot (which is satellite), with odd $\nu(K)$ [Greene-Wiest]. (Recall that the signature of knot is always even)
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(Remark: $n_K(0)$ is often the “hardest” value of $n_K$ to determine – recall the figure eight knot case.)
Main Results: Framing function and knot genus

**Proposition A**

For a knot $K$, let $g(K)$ be the Seifert genus of $K$. Then

$$n_K(0) \geq 2g(K)$$
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Actually, we have the following conjecture.

Conjecture A (I.)

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**Conjecture A (I.)**

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Our main result proves the conjecture for $g(K) = 1$.

**Main Theorem (conjectured by Greene-Wiest)**

If $K$ is not unknot, then

$$n_K(0) \geq 4$$
Dehn’s lemma

Theorem (Dehn’s lemma, Papakyriyakopoulos ’57)

Let $K$ be a knot (in $S^3$). Assume that there exists a map

$$f : D^2 \to S^3$$

which satisfies

1. $f(\text{Int} \ D^2) \cap K = \emptyset$.
2. $f|_{\partial D^2} = K$.
3. $f|_{N(\partial D^2)}$ is an embedding.

Then $K$ is unknot. Namely, there exists an embedding

$$f' : D^2 \to S^3$$

such that $f'|_{\partial D^2} = K$.

This result (and generalizations known as Loop theorem and Sphere theorem) are the most fundamental results in 3-dimensional topology.
Strengthened version of Dehn’s lemma

By using a language of compressing disc, Dehn’s lemma is equivalent to saying:

**Theorem (Dehn’s lemma)**

Let $K$ be a knot in $S^3$. If there exists a compressing disc $D$ of $K$ such that $i(D, K) = \#(K \cap D) = 0 \ (\iff n_K(0) = 0)$, then $K$ is unknot.

Then the Main Theorem says:

**Corollary (Strengthened version of Dehn’s lemma)**

Let $K$ be a knot in $S^3$. If there exists a compressing disc $D$ of $K$ such that $i(D, K) = 0$ and $\#(K \cap D) \leq 2 \ (\iff n_K(0) \leq 2)$, then $K$ is unknot.

Thus, we have a stronger form of Dehn’s lemma!
Strengthened version of Dehn’s lemma

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**Theorem (Dehn’s lemma)**

Let $K$ be a knot in $S^3$. If there exists a compressing disc $D$ of $K$ such that $i(D, K) = \#(K \cap D) = 0$ ($\iff n_K(0) = 0$), then $K$ is unknot.

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Thus, we have a stronger form of Dehn’s lemma!
Proof of Proposition: $n_K(0) \geq 2g(K)$

- Assume that $n_K(0) = m$ so we have an immersed compressing disc $D : D^2 \to S^3$
  
  
  with $i(D, K) = 0$ and $\#(D \cap K) = m$. 

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- Assume that $n_K(0) = m$ so we have an immersed compressing disc

\[ D : D^2 \to S^3 \]

with $i(D, K) = 0$ and $(D \cap K) = m$.

- $i(D, K) = 0$ means $m = 2n$ and the number of positive and negative intersections are the same.
Proof of Proposition: \( n_K(0) \geq 2g(K) \)

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- \( i(D, K) = 0 \) means \( m = 2n \) and the number of positive and negative intersections are the same.

- For a pair of positive and negative intersections, we attach a thin tube connecting them. This produces an immersed Seifert surface of genus \( n \).
Proof of Proposition (Continued)

Attach thin tubes to get immersed Seifert surface

Gabai’s theorem (immersed Seifert genus = usual (embedded) Seifert genus) says that $n \geq g(K)$. 
Proof of Theorem (Sketch –(i))

- Assume that $n_K(0) = 2$ but $K$ is not unknot.
  (By Proposition, $g(K) = 1$.)
Proof of Theorem (Sketch −(i))

- Assume that $n_K(0) = 2$ but $K$ is not unknot. (By Proposition, $g(K) = 1$.)
- From a compressing disc $D$ with $\#(D \cap K) = 2$, we get an immersed genus one Seifert surface

$$I : \Sigma_{1,1} \to S^3.$$ 

The co-core of tube $\gamma \subset I(\Sigma_{1,1})$ is not null-homologous:

$$\gamma \neq 0 \in H_1(S^3 - K).$$
Proof of Theorem (Sketch –(i))

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The co-core of tube \( \gamma \subset I(\Sigma_{1,1}) \) is not null-homologous:

\[
[\gamma] \neq 0 \in H_1(S^3 - K).
\]

**Strategy**

Prove \([\gamma] = 0\) in \( H_1(S^3 - K) \) by using topology and geometry of \( S^3 - K \). This leads to a contradiction.
Proof of Theorem (Sketch – (ii))

We “fill” the boundary, because

\[ \pi_1(\Sigma_{1,1}) = F_2 \text{ (Free group)}, \quad \pi_1(\text{Torus}) = \mathbb{Z}^2 \text{ (Abelian group)}. \]

- \( M \): Closed 3-manifold obtained 0-framed surgery along \( K \).
- \( F : \Sigma_{1,1} \hookrightarrow S^3 \): a genus one Seifert surface of \( K \).

Observations (i)

- \( F \) and \( I \) extends to an embedded/immersed torus \( bF \) and \( bI \) in \( M \).
- \( i : S^3 - K, \to M \) induces an isomorphism on the 1st homology group.

Thus, we actually try to show

**Modified Strategy**

We show \( i^* \big[ \big] = 0 \in H_1(M) \) to deduce a contradiction.
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Proof of Theorem (Sketch –(ii))

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Proof of Theorem (Sketch -(iii))

To show $i_*[\gamma] = 0 \in H_1(M)$,

What we actually prove

$\hat{F}$ and $\hat{I}$ can be put so that they are disjoint.

This is proven by looking intersection $\hat{F} \cap \hat{I}$. Then we use:

- Classification of centralizers (abelian subgroups) of (Haken) 3-manifolds.
  $\implies M$ must be very special kind of 3-manifold.

- Property of 0-framed surgery (Property R and related results)
  $\implies$ Such $M$ cannot be obtained by 0-surgery of genus one knot.
Open (tractable) questions

$n_K, \mathcal{L}(K), \nu(K)$ are less studied, so we have many, many open problems (and a lot of things to study)!

1. Try to make a table of $\nu(K)$ for 9–11 crossing knots.
2. Find a reasonable upper or lower bound by using other knot invariants. (Lower bounds seems to be more hard problem)
3. Try to compare signature $\sigma(K)$ and $\nu(K)$.
4. Try to find a compressing disc which is not a clasp disc. (i.e compare $n_K$ with clasp number)
5. Try to find a family of interesting (complicated, but having reasonable structure) compressing disc for several family of knots.
6. Find a sufficient conditions for $\nu(K) \neq 0$. 

(Remark: This paper is only 3 pages long!)