EXCEPTIONAL SURGERIES ON ALTERNATING KNOTS

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ABSTRACT. The famous Hyperbolic Dehn Surgery Theorem due to W. Thurston says that each hyperbolic knot admits only finitely many Dehn surgeries yielding non-hyperbolic manifolds. Concerning the maximal number of such exceptional surgeries, C. Gordon conjectured that there exist at most 10 for each knot. In this article, we report that this conjecture is true for hyperbolic alternating knots in the 3-sphere.

1. INTRODUCTION

As a consequence of the famous Geometrization Conjecture raised by W.P. Thurston in [24], all closed orientable 3-manifolds are classified as; reducible (i.e., containing essential 2-spheres), toroidal (i.e., containing essential tori), Seifert fibered (i.e., foliated by circles), or hyperbolic (i.e., admitting a complete Riemannian metric with constant sectional curvature $-1$). See [22] for a survey.

The recent Perelman’s works [18, 19, 20], where he announced an affirmative answer to this Geometrization Conjecture, is now going to become acceptable. Thus, in this article, we will assume that the Geometrization Conjecture can be affirmatively established; this implies that the above classification of closed orientable 3-manifolds can be achieved.

Beyond the classification, one of the next directions in the study of 3-manifolds would be to consider the relationships between 3-manifolds. One of the important operations describing relationships between 3-manifolds is Dehn surgery on a knot. That is an operation to create a new 3-manifold from a given one and a given knot (i.e., an embedded simple closed curve) in it as follows: Take an open tubular neighborhood of the knot, remove it, and glue a solid torus back. This gives an interesting subject to study; because, for example, it is known...
that any pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries on knots. It was proved by Lickorish [14] and Wallace [25] independently.

Another motivation to study Dehn surgery comes from the following famous fact, now called the Hyperbolic Dehn Surgery Theorem, due to W.P. Thurston [23]: On a hyperbolic knot (i.e., a knot with hyperbolic complement), all but finitely many Dehn surgeries yield hyperbolic 3-manifolds. In view of this, such finitely many exceptions are called exceptional surgeries. Then It is natural to ask:

**Question.** How many exceptional surgeries can occur on a knot?

Concerning this question, C.McA. Gordon conjectured that there exist at most 10 exceptional surgeries on each hyperbolic knot. See [12, Problem 1.77]. As far as the author knows, the sharpest known bound is; they are at most 12, which is obtained as a corollary of the so-called “6-theorem” given by Agol [1] and Lackenby [13] independently. We remark that, if we does not assume the Geometrization “Theorem”, then the best known is; at most 60, given by Hodgson and Kerckhoff [10].

2. **Result**

Our main result is the following.

**Main theorem.** On a hyperbolic alternating knot in the 3-sphere, there exist at most 10 exceptional surgeries.

Therefore the Gordon’s conjecture is true for such knots. Here a knot is called alternating if it admits a diagram with alternatively arranged over-crossings and under-crossings running along it.

Note that Gordon also conjectured that a hyperbolic knot with 10 exceptional surgeries must be the well-known figure-eight knot in the 3-sphere $S^3$ only. The figure-eight knot is also alternating, but our argument cannot tell that it is the only knot admitting 10 exceptional surgeries.

The theorem above follows from the next;

**Theorem 1.** On a hyperbolic alternating knot in the 3-sphere, all non-trivial exceptional surgeries are integral.

Together with the author’s previous work;
Theorem 2 (c.f., [11]). On any hyperbolic knot in the 3-sphere, there are at most 9 integral exceptional surgeries.

Here we recall fundamental terminologies. See [21] in details for example. As usual, by a slope, we call an isotopy class of a non-trivial unoriented simple closed curve on a torus. Then Dehn surgery on a knot $K$ is characterized by the slope on the peripheral torus of $K$ which is represented by the simple closed curve identified with the meridian of the attached solid torus via the surgery. When $K$ is a knot in $S^3$, by using the standard meridian-longitude system, slopes on the peripheral torus are parametrized by rational numbers with 1/0. For example, the meridian of $K$ corresponds to 1/0 and the longitude to 0. By the trivial Dehn surgery on $K$ in $S^3$, we mean the Dehn surgery on $K$ along the meridional slope 1/0. Thus it yields $S^3$ again, which is obviously exceptional. We say that a Dehn surgery on $K$ in $S^3$ is integral if it is along an integral slope. This means that the curve representing the slope runs longitudinally once.

3. Proof of Theorem 1

In this section, we give an outline of the proof of Theorem 1. As we are assuming that the Geometrization Conjecture is true, exceptional surgeries are divided into three types; reducible (i.e., yielding a reducible manifold), toroidal (i.e., yielding a toroidal manifold), or Seifert fibered (i.e., yielding a Seifert fibered manifold).

3.1. Reducible surgery on alternating knots. In [16], Menasco and Thistlethwaite studied essential surfaces (i.e., incompressible and not boundary-parallel surfaces) properly embedded in alternating knot exteriors, and established that no Dehn surgeries on a hyperbolic alternating knot in $S^3$ yield reducible manifolds. In fact, F. González-Acuña and H. Short conjectured in [8] that the only way to get a reducible 3-manifold by surgery on a knot in $S^3$ is to surger on a cable knot along the slope determined by the cabling annulus. This is now called the Cabling Conjecture, and still remaining open. See [12, Problem 1.79].

3.2. Toroidal surgery on alternating knots. In [17], based on the result obtained in [16], Patton showed that only integral surgeries on a hyperbolic alternating knot in $S^3$ yield toroidal manifolds. In fact, in [9], Gordon and Luecke proved that if a hyperbolic knot in $S^3$ admits a non-integral toroidal surgery, then all the knots are ones explicitly given by Eudave-Muñoz in [5].
3.3. **Seifert fibered surgery on alternating knots.** By virtue of these results, from now on, we only consider Seifert fibered surgery on alternating knots.

In [13], Lackenby studied exceptional surgeries on alternating knots, and give the following: If a hyperbolic alternating knot $K$ has a prime alternating diagram $D$ satisfies $t(D) > 4$, then only integral surgeries on $K$ yield Seifert fibered manifolds. Here $t(D)$ denotes the *twist number* of the diagram $D$. That is, the number of *twists*, which are either; maximal connected collections of bigon regions in the complement of $D$ arranged in a row or isolated crossings adjacent to no bigon regions.

In the case where $t(D) \leq 4$, using elementary diagrammatic arguments, we have the following:

**Lemma.** Suppose that a hyperbolic alternating knot $K$ has a prime alternating diagram $D$ with $t(D) \leq 4$. Then $K$ is either; a two-bridge knot, an arborescent knot of type III, or a Montesinos knot of length 3.

Thus we can divide the arguments into three cases as follows.

3.3.1. **Seifert fibered surgery on two-bridge knots.** A *bridge index* of a knot in $S^3$ is defined as the minimal number of local maxima (or local minima) up to ambient isotopy. Thus a knot with bridge index 2 is called a *two-bridge knot*. Note that Menasco showed in [15] that a two-bridge knot is hyperbolic unless it is a $(2, p)$-torus knot.

In [3], Brittenham and Wu gave a complete classification of the exceptional surgeries on hyperbolic two-bridge knots. From this, we can verify that; on a hyperbolic two-bridge knot, only integral surgeries yield Seifert fibered manifolds.

We remark that the key ingredient in their proof is using *essential laminations* in 3-manifolds, defined by Gabai and Oertel in [7] as follows: We say a lamination $\lambda$ (i.e., a co-dimension one foliation of a closed subset of the ambient manifold) is an *essential lamination* in a 3-manifold $M$ if it satisfies the following conditions:

(i) The inclusion of leaves of $\lambda$ into $M$ induces an injection between their fundamental groups.

(ii) The complement of $\lambda$ is irreducible.

(iii) The lamination $\lambda$ has no sphere leaves.
(iv) The lamination $\lambda$ is end-incompressible.
About essential laminations, see [6] for example.

3.3.2. **Seifert fibered surgery on arborescent knots.** In [26], also using essential laminations mainly, Y.-Q. Wu showed that; an arborescent knot of type III has no exceptional surgeries. In particular, there are no surgeries yielding Seifert fibered manifolds.

Here we recall definitions of an arborescent knot and its type. See [26] or [27] for details.

By a **tangle**, we mean a pair of a 3-ball and properly embedded arcs. From two arcs of rational slope drawn on the boundary of a pillowcase-shaped 3-ball, one can obtain a tangle, which is called a *rational tangle*. A tangle obtained by putting rational tangles together in a horizontal way is called a *Montesinos tangle*. An *arborescent tangle* is then defined as a tangle that can be obtained by summing several Montesinos tangles together in an arbitrary order.

Suppose that a knot $K$ in $S^3$ is obtained by closing a tangle $T$. If $T$ is a Montesinos tangle, then we call $K$ a *Montesinos knot*, and if $T$ is an arborescent tangle, then we call $K$ an *arborescent knot*. For a Montesinos knot, the number of rational tangles forming the corresponding Montesinos tangle is called the *length* of the Montesinos knot.

In [26], Wu divide all arborescent knots into three types: By the type I knots, we mean two-bridge knots or Montesinos knots of length 3. A knot of type II is defined as the union of two Montesinos tangles, each of which is formed by two rational tangles corresponding to $1/2$ and an integer. All other arborescent knots are called of type III.

3.3.3. **Seifert fibered surgery on Montesinos knots.** In the remaining case: for hyperbolic alternating Montesinos knots of length 3, we can prove the following proposition.

**Proposition.** On an alternating Montesinos knot of length 3, only integral surgeries yield Seifert fibered manifolds.

Also in this case, in an unpublished preprint [4], Delman gave a construction of essential lamination in the knot exterior. By examining his construction, we can verify that each essential lamination so constructed admits two disjoint, nonparallel annuli properly embedded in the complement of the lamination satisfying that; one boundary component is the meridian of the knot and the
other lies in some leaf of the lamination. Having been suggested in [2, 27], the existence of such a pair of annuli guarantees that non-integral Dehn surgery on the knot never yields a Seifert fibered manifold. Precisely, in that case, the lamination so constructed in the knot exterior survives essential and becomes genuine via any non-integral Dehn surgeries. This implies that the resultant manifold cannot have a finite fundamental group, by definition, and is not a small Seifert fibered manifold shown in [2].

Combining these, we complete the proof of Theorem 1.

**References**


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