Complete classifications of exceptional surgeries on Montesinos knots and alternating knots

Kazuhiro Ichihara

Nihon University
College of Humanities and Sciences

joint work with

In Dae Jong (Kinki U.), Hidetoshi Masai (U. of Tokyo)

SEOUL ICM 2014
Coex, Seoul, Korea, August 18, 2014
Classification of 3-manifolds

Every closed orientable 3-manifold is;

- **Reducible** (containing essential 2-sphere),
- **Toroidal** (containing essential torus),
- **Seifert fibered** (foliated by circles), or
- **Hyperbolic** ($\exists$ Riem. metric of curv.$\equiv-1$).

as a consequence of the **Geometrization Conjecture**

including famous **Poincaré Conjecture** (1904)
conjectured by Thurston (late ’70s)
established by Perelman (2002-03)
### Motivation

**Hyperbolic Dehn Surgery Theorem [Thurston (1978)]**

Only **finitely many** Dehn surgeries on a **hyperbolic** knot (i.e., knot with hyperbolic complement) yield **non-hyperbolic** manifolds.

**Exceptional surgery**

A Dehn surgery on a **hyperbolic** knot is called **exceptional** if it yields a **non-hyperbolic** manifold.

**Ultimate Goal**

Classify all the exceptional surgeries on hyperbolic knots in the 3-sphere $S^3$.

**Remark:** “exceptional” $\Leftrightarrow$ reducible, toroidal, Seifert fibered
Results: Alternating knots

Theorem [I.-Jong (Proc Japan Acad, 2014)]

Alternating knots admit no toroidal Seifert surgeries other than the trefoil knot or some composite knots.


Let $K$ be a hyperbolic alternating knot in $S^3$. If $K$ admits a non-trivial exceptional surgery, then $K$ is equivalent to an arborescent knot.

$\Rightarrow$ Complete classification!
Base code

To study exceptional surgeries on links, we basically used a computer program developed in:

B. Martelli, C. Petronio, F. Roukema
Exceptional Dehn surgery on the minimally twisted five-chain link
preprint, arXiv:1109.0903v1

The program relies upon

- SnapPy (based on SnapPea): computer software calculates various hyperbolic invariants for 3-manifolds.
  http://www.math.uic.edu/t3m/SnapPy/
Key Ingredients

We modified the original codes to use interval arithmetics and applied the program \texttt{hikmot} developed in

Hoffman, Ichihara, Kashiwagi, Masai, Oishi, and Takayasu

**Verified computations for hyperbolic 3-manifolds**


http://www.oishi.info.waseda.ac.jp/~takayasu/hikmot/

It can possibly give us a rigorous complete classification of exceptional surgeries on a given hyperbolic link.
Computation time

- We have 30404 links to investigate.

- Our code applies hikmot recursively.

  In the worst case, for a single link, there are about 18,000 manifolds to investigate.

  ⇒ It takes around 51 HOURs by single CPU.

  We need a high spec machine!!
TSUBAME

- TSUBAME is the supercomputer of Tokyo Tech. providing large-scale parallel computing.

In total, i.e. the sum of the computation time of all nodes, computation time was approximately 512 days, and the number of manifolds we applied hikmot is 5,646,646.
Results: Montesinos knots

**Theorem 7** [I.-Masai, 2013 (arXiv:1310.3472)]

The Montesinos knots $M(-1/2, 2/5, 1/(2q + 1))$ with $q \geq 5$ have no non-trivial exceptional surgeries.

Together with known results, this gives the final step to establish the **complete classification** of exceptional surgeries on **arborescent knots** (including **Montesinos knots**).

Let $K$ be a hyperbolic alternating knot in $S^3$. Suppose that $K(r)$ is non-hyperbolic for a rational number $r$.

- $K(r)$ is irreducible [Menasco-Thistlethwaite (’92)]
- $r \in \mathbb{Z}$ [I. AGT (’08)]
- If $K(r)$ is toroidal, then $K(r)$ is not a SF [I.-J., PJA (’14)], and $K$ is equivalent to either
  - the figure-eight knot and $r = 0, \pm 4$,
  - a two bridge knot $K_{[b_1,b_2]}$ with $|b_1|, |b_2| > 2$, and $r = 0$ if both $b_1, b_2$ are even, $r = 2b_2$ if $b_1$ is odd and $b_2$ is even,
  - a twist knot $K_{[2n,\pm 2]}$ with $|n| > 1$ and $r = 0, \mp 4$,
  - a pretzel knot $P(q_1,q_2,q_3)$ with $q_i \neq 0, \pm 1$, and $r = 0$ if $q_1, q_2, q_3$ are all odd, $r = 2(q_2 + q_3)$ if $q_1$ is even and $q_2, q_3$ are odd.

[Boyer-Zhang (’97)], [Patton (’95)]

- If $K(r)$ is small Seifert fibered, then $\pi_1(K(r))$ is infinite [Delman-Roberts ('99)] and $K$ is equivalent to either
  - the figure-eight knot and $r = \pm 1, \pm 2, \pm 3$,
  - a twist knot $K_{2n, \pm 2}$ with $|n| > 1$ and $r = \mp 1, \mp 2, \mp 3$.

In particular, the figure-eight knot is the only knot admitting 10 exceptional surgeries among hyperbolic alternating knots, and the others admit at most 5 exceptional surgeries.
Classification II

Based on:
[Brittenham-Wu ('01)], [Wu ('11,'11,'12)],
[I.-Jong ('13)], [Meier ('14)]


Let $K$ be a hyperbolic arborescent knot in $S^3$. Suppose that $K(r)$ is non-hyperbolic for $r \in \mathbb{Q}$.

Then $r$ must be an integer except for $r = 37/2$ for $P(-2, 3, 7)$.

The manifold $K(r)$ is always irreducible [Wu ('96)], and $\pi_1(K(r))$ is infinite except for $r = 17, 18, 19$ for $P(-2, 3, 7)$ and $r = 22, 23$ for $P(-2, 3, 9)$.

[I.-Jong, AGT ('09)]

If $K(r)$ is toroidal, then $K(r)$ is not a Seifert fibered, [I.-Jong, CAG ('10)] and $K$ is equivalent to

- a two bridge knot $K_{[b_1,b_2]}$ with $|b_1|, |b_2| > 2$, and $r = 0$ if both $b_1, b_2$ are even, $r = 2b_2$ if $b_1$ is odd and $b_2$ is even,
- a twist knot $K_{[2n,\pm2]}$ with $|n| > 1$ and $r = 0, \pm4$,
- one of the Montesinos knots of length 3 with the slope described in Table 1.

- $K_1$ with $r = 3$, $K_2$ with $r = 0$ or $K_3$ with $r = -3$ for
  \begin{align*}
  (S^3, K_1) &= T(1/3, -1/2; 4) \cup_{\eta} T(1/3, -1/2; 4), \\
  (S^3, K_2) &= T(1/3, -1/2; 4) \cup_{\eta} T(-1/3, 1/2; -4), \text{ and} \\
  (S^3, K_3) &= T(-1/3, 1/2; -4) \cup_{\eta} T(-1/3, 1/2; -4).
  \end{align*}
## Classification II

**Table:** Toroidal surgeries

<table>
<thead>
<tr>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(q_1, q_2, q_3)$, $q_i$ odd and $</td>
<td>q_i</td>
</tr>
<tr>
<td>$P(q_1, q_2, q_3)$, $q_1$ even, $q_2, q_3$ odd and $</td>
<td>q_i</td>
</tr>
<tr>
<td>$P(-2, 3, 7)$</td>
<td>$37/2$</td>
</tr>
<tr>
<td>$P(-3, 3, 7)$</td>
<td>1</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(3 + 1/n))$, $n$ even and $n \neq 0$</td>
<td>$2 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(5 + 1/n))$, $n$ even and $n \neq 0$</td>
<td>$1 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(6 + 1/n))$, $n \neq 0$, $-1$ odd (resp. even)</td>
<td>$16$ (resp. 0)</td>
</tr>
<tr>
<td>$M(-1/2, 1/5, 1/(3 + 1/n))$, $n$ even and $n \neq 0$</td>
<td>$5 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 2/5, 1/7)$</td>
<td>12</td>
</tr>
<tr>
<td>$M(-1/2, 2/5, 1/9)$</td>
<td>15</td>
</tr>
<tr>
<td>$M(-1/3, -1/(3 + 1/n), 2/3)$, $n \neq 0$, $-1$ odd (resp. even)</td>
<td>$-12$ (resp. 4)</td>
</tr>
<tr>
<td>$M(-2/3, 1/3, 1/4)$</td>
<td>13</td>
</tr>
<tr>
<td>$M(-1/(2 + 1/n), 1/3, 1/3)$, $n$ odd and $n \neq -1$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

If $K(r)$ is small Seifert fibered, then $K$ is either

- the figure-eight knot and $r = \pm 1, \pm 2, \pm 3$,
- a twist knot $K_{2n, \pm 2}$ with $|n| > 1$ and $r = \mp 1, \mp 2, \mp 3$,
- one of the Montesinos knots of length 3 with the slope described in Table 2.
### Classification II

**Table:** Seifert fibered surgeries

<table>
<thead>
<tr>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(-2, 3, 2n + 1), n \neq 0, 1, 2$</td>
<td>$4n + 6$, $4n + 7$</td>
</tr>
<tr>
<td>$P(-2, 3, 7)$</td>
<td>17</td>
</tr>
<tr>
<td>$P(-3, 3, 3)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 4)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 5)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 6)$</td>
<td>1</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/5)$</td>
<td>3, 4, 5</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/7)$</td>
<td>$-1, 0, 1$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/9)$</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/11)$</td>
<td>$-1, -2$</td>
</tr>
<tr>
<td>$M(-1/2, 1/5, 2/5)$</td>
<td>7, 8</td>
</tr>
<tr>
<td>$M(-1/2, 1/7, 2/5)$</td>
<td>11</td>
</tr>
<tr>
<td>$M(-2/3, 1/3, 2/5)$</td>
<td>$-5$</td>
</tr>
</tbody>
</table>
Thank you very much