Obstructions to Quasi-alternating links

Nafaa Chbili (United Arab Emirates University)*

1. Introduction

Alternating links represent an important class of links in the three-sphere which admit a simple diagrammatic definition. They have been subject to extensive literature. In particular, the study of their Jones polynomials led to the proof of old challenging conjectures in knot theory, [18]. Their Khovanov homologies are quite simple and depend only on the Jones polynomial and the signature of the link, [12]. Similarly, their link Floer homologies are determined by the Alexander polynomial and the signature of the link [21]. In addition, the Heegaard Floer Homology of their branched double covers Σ_L depends only on the determinant of the link, $det(L)$. In this context, an interesting generalization of alternating links has been obtained by Ozsváth and Szabó, [21]. They proved that this homological property of Σ_L extends to a wider class of links, that they called quasi-alternating links. Unfortunately, unlike alternating links which admit a simple definition, quasi-alternating links are defined in a recursive way. The recursive nature of the definition makes it uneasy to decide whether a given link is quasi-alternating by the only use of the definition. Several obstructions criteria have been developed to characterize quasi-alternating links. The purpose of this paper is to give a kind of short introduction to the subject, focusing on the obstructions criteria introduced recently in terms of quantum invariants of links. The paper is based on the work of Qazaqzeh and Chbili [22] and Teragaito [28, 29].

2. Alternating Links

An link diagram is said to be alternating if the underpass and the over pass alternate when one travels along any component of the diagram. The alternating diagram is reduced if it contains no isthmus. A link L is alternating if it admits an alternating diagram. The Jones polynomial $V_L(t)$ is an invariant of ambient isotopy of oriented links in the three-sphere. This invariant admits several equivalent definitions. The simplest is the following recursive definition on link diagrams:

$$V_U(t) = 1,$$
$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) = (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}(t),$$

where U is the unknot and L_+, L_- and L_0 are three links which are identical except in a small ball where they are as pictured in Figure 1. Kauffman introduced a two-variable generalization of the Jones polynomial which can be defined as follows. Let $\Lambda(a, z)$ be the invariant of regular isotopy of un-oriented link diagrams defined by the following relations:

$$\Lambda_{L_+}(z) + \Lambda_{L_-}(z) = z(\Lambda_{L_0}(z) + \Lambda_{L_\infty}(z))$$
$$\Lambda = a\Lambda$$
$$\Lambda = a^{-1}\Lambda$$

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*e-mail: nafaachbili@uaeu.ac.ae

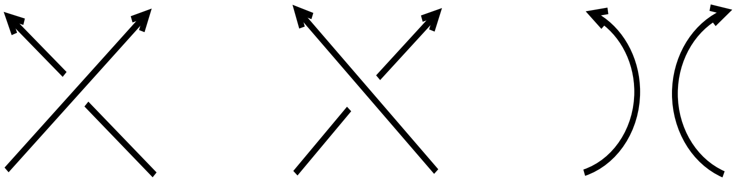

Figure 1: L_+, L_- and L_0, respectively.

where L_+, L_-, L_0 and L_∞ are four links which are identical except in a small ball where they are as in the following picture:

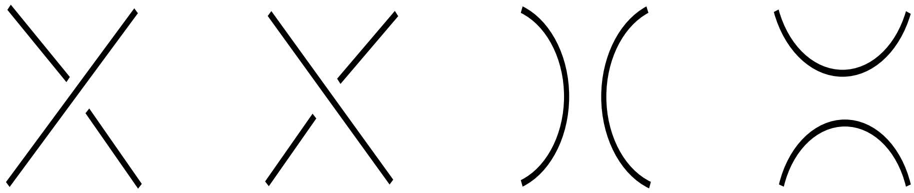

Figure 2: L_+, L_-, L_0 and L_∞, respectively.

Let D is an oriented link diagram of a link L and $w(D)$ its writhe. Then, the two-variable Kauffman polynomial of L is defined by $F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$, where Λ_D is obtained by forgetting the orientation of D. The polynomial F_L is an invariant of ambiant isotopy of links which specializes to the Jones polynomial. Another interesting specialization of the Kauffman polynomial is the Brandt-Lickorish-Millet polynomial $Q(x)$, known also as the Q-polynomial [2]. This invariant was introduced shortly after the discovery of the Jones polynomial and it can be obtained from the Kauffman polynomial as $Q(x) = F(1,x)$.
For an oriented link L, we define the determinant of L by $det(L) = |V_L(-1)|$. This numerical invariant of links was first defined from the Seifert Matrix. It is well known that $det(L) = \sqrt{Q(2)}$ and that if Σ_L is the branched double cover of L, then $det(L)$ is equal to the order of the first homology group of Σ_L. If a link L is alternating with an alternating connected diagram D, then $det(L)$ is known to be equal to the number of spanning trees in the Tait graph associated with L. The breadth of the Jones polynomial of an oriented link $breadthV_L(t)$ is defined to be the difference between the highest and the lowest power of t that appear in $V_L(t)$. We denote by $degQ_L$ the highest power that appears in $Q_L(x)$ and by deg_zF_L the highest power of the variable z that appears in $F_L(a,z)$. We have $0 \leq degQ_L \leq deg_zF_L$. Finally, for any link L we denote by $\sigma(L)$ the signature of L. It was shown that the Jones polynomial of alternating links satisfies the following properties [27].
Theorem 2.1 [27]. If L is a non split alternating link, then:

1. $breadthV_L = c(L)$, where $c(L)$ is the crossing number of L.

2. $V_L(t) = \sum_i a_i t^i$ is an alternating polynomial.

3. The coefficients of the highest and lowest degree in $V_L(t)$ are both ∓ 1.

These conditions on the Jones polynomials of alternating links led to the solution of two of the most intriguing questions in classical knot theory asked by Tait in the nineteenth century. In particular, that a reduced alternating knot diagram has the minimal number of crossings and that its writhe is an invariant of the knot.

In 1999, Khovanov introduced a bi-graded link homology theory whose Euler characteristic is the Jones polynomial [10]. Let L be an l-component oriented link and $KH^{*,*}$ be its reduced Khovanov homology, then $V_L(q) = \sum_{i\in\mathbb{Z}, j\in\mathbb{Z}+\frac{l-1}{2}} (-1)^i q^j rank(KH^{i,j}(L))$.
Another link homology theory $HFK^{*,*}$ called link Floer homology was introduced by Ozsváth and Szabó in 2002. For any link L, the graded Euler characteristic of $HFK^{*,*}(L)$ is the Alexander polynomial of L.
If L is an alternating link, then the Khovanov homology of L is determined by the Jones polynomial $V_L(t)$ and the signature of the link, $\sigma(L)$. Similarly, the link Floer homology of an alternating link depends only on the Alexander polynomial and the signature of the link. More precisely, these links are homologically thin in both Khovanov and link Floer homologies. For instance, $KH^{i,j}(L)$ is trivial whenever $i - j \neq \frac{\sigma(L)}{2}$.

3. Quasi-alternating Links

Ozsváth and Szabó studied the Heegaard Floer homology of the branched double cover Σ_L of S^3 branched over a link L [21]. They proved that if L is non split alternating link then the homology of Σ_L is determined by the determinant of the link. Rational homology spheres with such simple Heegaard Floer homology are called L-spaces. Examples of such 3-manifolds include lens spaces, and Seifert fibered manifolds with finite fundamental group. In the same paper, it was proved that these homological properties extend to a wider class of links, which the authors call quasi-alternating links. These links are defined recursively as follows.
Definition 3.1. *The set $\mathcal{Q}$ of quasi-alternating links is the smallest set satisfying the following properties:*

1. *The unknot belongs to $\mathcal{Q}$.*
2. *If L is a link with a diagram D containing a crossing c such that*
   (a) *both smoothings of the diagram D at the crossing c, L_0 and L_∞ as in Figure 3 belong to $\mathcal{Q}$;*
   (b) $\det(L_0), \det(L_\infty) \geq 1$;
   (c) $\det(L) = \det(L_0) + \det(L_\infty)$;
   then L is in $\mathcal{Q}$ and in this case we say that L is quasi-alternating at the crossing c with quasi-alternating diagram D.

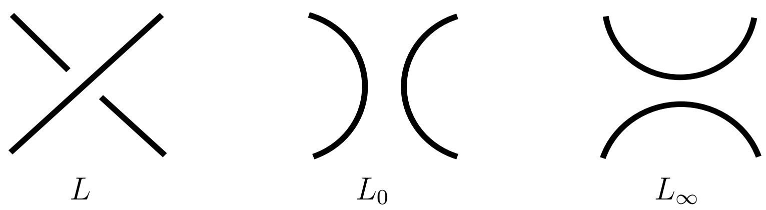

Figure 3: The diagram of the link L at the crossing c and its smoothings L_0 and L_∞ respectively.

Here are some basic facts about this class of links. These facts can be obtained easily from the definition and an elementary induction on the determinant of the link.

- The determinant of a quasi-alternating link is always positive and it is equal to 1 if and only if L is the unknot.
- Any non-split alternating link is quasi-alternating at any crossing of any reduced alternating diagram [21].
- If K_1 and K_2 are two quasi-alternating knots, then so is their connected sum $K_1 \sharp K_2$, [3].
- If L is quasi-alternating, then so is its mirror image $L!$.

Using the definition, it was shown that the non-alternating knot 9_{47} is quasi-alternating by giving a quasi-alternating 11-crossing diagram of the knot, see [21]. A simple way to produce new examples of quasi-alternating links from old ones was introduced by Champanerkar and Kofman [3]. Given a link L with quasi-alternating diagram D at a crossing c. Then any link diagram obtained by replacing the crossing c by an alternating rational tangle of the same type is quasi-alternating at any of the new crossings. This construction was applied to prove the quasi-alternateness of the Pretzel links of type $P(p_1, \ldots, p_n, -q)$, where $n \geq 1$, $p_i \geq 1$ for all i, and $q \geq min\{p_1, \ldots, p_n\}$. Quasi-alternating links with determinant at most 7 are known to be either two-bridge links or connected sums of two-bridge links, see [13]. Consequently, if L is a non alternating quasi-alternating link then $det(L) \geq 8$.
Ozsváth and Szabó proved that if L is quasi-alternating, then Σ_L is an L-space which bounds a negative definite $4-$manifold W with $H_1(W) = 0$. The Khovanov and link Floer homologies of quasi-alternating links have been computed in [16].

Theorem 3.2 [16]
If L is a quasi-alternating link, then:
(i) The reduced Khovanov Homology is σ-thin (over $\mathbb{Z}$);
(ii) The link Floer homology is σ-thin (over $\mathbb{Z}_2$);

It is worth mentioning here that the previous theorem extends to odd Khovanov homology as well [19]. Among the 85 knots with up to 9 crossings only the two knots 8_{19}, which is indeed the $(3,4)$-torus knot, and 9_{42} don't satisfy the conditions of the previous theorem. So they are not quasi-alternating. The knot 9_{46} is odd Khovanov homology thick, hence not quasi-alternating. All the other 82 knots are quasi-alternating, see [1, 15, 21]. Among these 82 knots there are only 8 which are quasi-alternating non-alternating knots:

$$8_{20}, 8_{21}, 9_{43}, 9_{44}, 9_{45}, 9_{47}, 9_{48}, 9_{49}.$$

The list of quasi-alternating knots with 10-crossings has been also determined. In addition to alternating knots, the list includes 31 non-alternating knots, see [3].
For $p, q \geq 3$, the $(p,q)-$torus knot is Khovanov homology thick, so not quasi-alternating [26]. So are adequate non-alternating knots [11]. In general, these homological properties techniques have been of great help in the characterization of quasi-alternating links. However, Greene proved the existence of homologically-thin non quasi-alternating knots by showing that the knot 11_{n50} is homologically-thin in reduced Khovanov, odd Khovanov and link Floer homology but not quasi-alternating [5]. An infinite family of

homologically thin, hyperbolic non quasi-alternating knots is given in [7]. Different other techniques have been used to find obstructions to quasi-alternateness of links and led to interesting results. In particular:
Quasi alternating Pretzel links have been determined, see [3, 5]. An extension to Montesinos links has been discussed in [4, 23, 30].
Quasi-alternating links of braid index at most 3 have been determined by Baldwin [1] based on Murasugi's classification of 3-braids [17].

4. Polynomial invariants of quasi-alternating links

The way the Jones polynomial interacts with the alternateness of links suggests that such polynomials invariants might be useful for the study of quasi-alternating links as well. Since the Jones polynomial is obtained as the Euler characteristic of the Khovanov homology, then the $\sigma-$thinness of this homology implies that the Jones polynomial of a quasi-alternating link is alternating [11]. By studying the relationship between the degree of the Q-polynomials of the quadruplet L_+, L_-, L_0 and L_∞, Qazaqzeh and Chbili [22] proved the following:

Theorem 4.1 [22]. *If L is a quasi-alternating link, then:* $\deg Q_L \leq \det(L) - 1$.

A slightly more precise condition has been obtained by Teragaito [28], as in the following theorem:

Theorem 4.2 [28]. *Let L be a quasi-alternating link, then either:*
(i) L is the $(2, n)$ torus link, or
(ii) $\deg Q_L < \det(L) - 1$.

These conditions are easy to apply since they relay only on the computation of the $Q-$polynomials. They can be used to rule out the quasi-alternateness of several knots and links. By considering the figure eight knot, Teragaito showed that the condition given by Theorem 4.2 is sharp. A nice application of Theorem 4.1 shows that there are only finitely many Kanenobu knots which are quasi-alternating. The Kanenobu knot $K(p, q)$ is known to have determinant equal to 25 regardless of the values of the two integers p and q. According to [24], the degree of $Q_{K(p,q)}$ is $|p| + |q| + 6$ if $pq \geq 0$ and $|p| + |q| + 5$ otherwise. Thus, if $K(p, q)$ is quasi-alternating then $|p| + |q| < 19$. This gives a simple proof to the non quasi-alternatness of the interesting family of homologically thin knots introduced in [7].

Teragaito proved a similar necessary condition for quasi-alternateness using the two-variable Kauffman polynomial:
Theorem 4.3 [29]. *Let L be a quasi-alternating link, then either:*
(i) L is a $(2; n)$-torus link for $n \neq 0$, and $deg_z F_L = det(L) - 1$;
(ii) L is the figure-eight knot or the connected sum of two Hopf links, and $deg_z F_L = det L - 2$; or
(iii) $deg_z F_L \leq det(L) - 3$.

These results are stronger than the ones obtained in terms of the $Q-$polynomials. Teragaito gave an infinite family of links where the condition on $deg Q_L$ fails to decide on the quasi-alternateness of the link, while the condition on $deg_z F_L$ does. As a consequence of the results above, Teragaito determined all the quasi-alternating links with determinant less than 5, [29]. Later on, Lidman and Sivek determined all quasi-alternating links with determinant less than 8, [13].

We close this discussion with the following conjecture about the Jones polynomial of quasi-alternating links. This conjecture is obviously true for non-split alternating links. It was also proved to be true in the case of links with braid index at most 3, see [22]. A more general conjecture has been discussed in [25].

Conjecture 4.4. *If L is a quasi-alternating link, then: $breadth V_L \leq det(L)$.*

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