On the minimal coloring number of even-parallels of links

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Let $L$ be a link, and $D$ a diagram of $L$.

**$\mathbb{Z}$-coloring**

A map $C : \{\text{arcs of } D\} \rightarrow \mathbb{Z}$ is called a **$\mathbb{Z}$-coloring** on $D$ if it satisfies the condition $2C(a) = C(b) + C(c)$ at each crossing of $D$ with the over arc $a$ and the under arcs $b$ and $c$.

A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called the **trivial $\mathbb{Z}$-coloring**.

**$\mathbb{Z}$-colorable link**

$L$ is **$\mathbb{Z}$-colorable** if $\exists$ a diagram of $L$ with a non-trivial $\mathbb{Z}$-coloring.
Let $L$ be a $\mathbb{Z}$-colorable link.

**Minimal coloring number**

We define the **minimal coloring number** of $L$, denoted by $\text{mincol}_\mathbb{Z}(L)$, as follows.

$$\min\{\#\text{Im}(C) \mid C : \text{non-trivial } \mathbb{Z}\text{-coloring on a diagram of } L\}$$
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**Theorem [Ichihara-M.]**

Let $L$ be a non-splittable $\mathbb{Z}$-colorable link. If there exists a simple $\mathbb{Z}$-coloring on a diagram of $L$, then $\mincol_{\mathbb{Z}}(L) = 4$.

**Theorem [Ichihara-M.]**

If a non-splittable link $L$ admits a $\mathbb{Z}$-coloring $C$ such that $\#\text{Im}(C) = 5$, then $\mincol_{\mathbb{Z}}(L) = 4$. 
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**Question**

For any $\mathbb{Z}$-colorable link $L$, $\mincol_{\mathbb{Z}}(L) = 4$?
Parallel of a link

For a link $L = K_1 \cup \cdots \cup K_c$ with a diagram $D$ and a set $(n_1, \cdots, n_c)$ of integers $n_i \geq 1$, we denote by $D^{(n_1, \cdots, n_c)}$ the diagram obtained by taking $n_i$-parallel copies of the $i$-th component $K_i$ of $D$ on the plane for $1 \leq i \leq c$. The link $L^{(n_1, \cdots, n_c)}$ represented by $D^{(n_1, \cdots, n_c)}$ is called the $(n_1, \cdots, n_c)$-parallel of the link $L$.

When $L$ is a knot, we call $(n)$-parallel $L^{(n)}$ simply an $n$-parallel, and denote it by $L^n$. 
Untwisted 2-parallel

A 2-parallel $K^2 = K_1 \cup K_2$ of a knot $K$ is called the untwisted 2-parallel where $\text{lk}(K_1, K_2) = 0$. 
Theorem 1
The untwisted 2-parallel $K^2$ of a knot $K$ is $\mathbb{Z}$-colorable and $\text{mincol}_\mathbb{Z}(K^2) = 4$.

Theorem 2
For any diagram of a $c$-component link $L$ and any even number $n_1, \cdots, n_c$ at least 4, $L^{(n_1, \cdots, n_c)}$ is $\mathbb{Z}$-colorable and $\text{mincol}_\mathbb{Z}(L^{(n_1, \cdots, n_c)}) = 4$. 
Outline of proof of Theorem 2

Let \( L = K_1 \cup \cdots \cup K_c \) be a link, and \( D \) a diagram of \( L \). We focus on crossings on \( D^{(n_1, \cdots, n_c)} \) obtained by taking parallel copies at a crossing of \( D \).
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For any parallel arcs \((a_1, \cdots, a_k)\) out of the circle, we fix the colors of \(a_k/2\) and \(a_k/2+1\) are 1 and others are 0.
For any arcs inside the circle, we assign colors as follows.

In the case \( n_j = 4m + 2 (m \in \mathbb{N}) \), we assign the colors \(-1, 0, 1, 2\).
In the case $n_j = 4m + 4 (m \in \mathbb{N})$, we assign the colors $-1, 0, 1, 2, 3$.  

\[ n_i \left\{ \begin{array}{c} 0 \quad 0 \\ 0 \quad 0 \\ \vdots \\ 0 \quad 0 \\ 1 \quad -1 \\ 1 \quad -1 \\ \vdots \\ 0 \quad 0 \end{array} \right\} \ 
\left\{ \begin{array}{c} 0 \quad 0 \\ 0 \quad 0 \\ \vdots \\ 0 \quad 0 \\ 1 \quad -1 \\ 1 \quad -1 \\ \vdots \\ 0 \quad 0 \end{array} \right\} \ 
\left\{ \begin{array}{c} 2 \quad 0 \\ 2 \quad 0 \\ \vdots \\ 2 \quad 0 \\ 1 \quad -1 \\ 1 \quad -1 \\ \vdots \\ 0 \quad 0 \end{array} \right\} \ 
\left\{ \begin{array}{c} 0 \quad 0 \\ 0 \quad 0 \\ \vdots \\ 0 \quad 0 \\ 1 \quad -1 \\ 1 \quad -1 \\ \vdots \\ 0 \quad 0 \end{array} \right\} \ 
\left\{ \begin{array}{c} 0 \quad 0 \\ 0 \quad 0 \\ \vdots \\ 0 \quad 0 \\ 1 \quad -1 \\ 1 \quad -1 \\ \vdots \\ 0 \quad 0 \end{array} \right\} \ 
\left\{ \begin{array}{c} 2m \\ n_j \\ 2m \end{array} \right\} \]
We see that $D^{(n_1, \cdots, n_c)}$ admits a $\mathbb{Z}$-coloring $C$ such that $\text{Im}(C) = \{-1, 0, 1, 2, 3\}$. Therefore $L^{(n_1, \cdots, n_c)}$ is $\mathbb{Z}$-colorable.

Moreover, we eliminate the arcs colored by 3 as follows.

It follows $\text{mincol}_\mathbb{Z}(L^{(n_1, \cdots, n_c)}) = 4$. □
Thank you for your attention.