On the minimal coloring number of even-parallels of links

Eri Matsudo

Nihon University
Graduate School of Integrated Basic Sciences

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$\mathbb{Z}$-coloring

Let L be a link, and D a diagram of L.

$\mathbb{Z}$-coloring

A map $C : \{\text{arcs of } D\} \to \mathbb{Z}$ is called a $\mathbb{Z}$-coloring on D if it satisfies the condition $2C(a) = C(b) + C(c)$ at each crossing of D with the over arc a and the under arcs b and c.

A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called the trivial $\mathbb{Z}$-coloring.

$\mathbb{Z}$-colorable link

L is $\mathbb{Z}$-colorable if $\exists$ a diagram of L with a non-trivial $\mathbb{Z}$-coloring.

Let L be a $\mathbb{Z}$-colorable link.

Minimal coloring number

We difine the minimal coloring number of L, denoted by $mincol_{\mathbb{Z}}(L)$, as follows.

$$\min\{\#\mathrm{Im}(C) \mid C : \text{non-trivial } \mathbb{Z}\text{-coloring on a diagram of } L\}$$

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Theorem [Ichihara-M.]

Let L be a non-splittable $\mathbb{Z}$-colorable link. If there exists a simple $\mathbb{Z}$-coloring on a diagram of L, then $mincol_{\mathbb{Z}}(L) = 4$.

Theorem [Ichihara-M.]

If a non-splittable link L admits a $\mathbb{Z}$-coloring C such that $\#\mathrm{Im}(C) = 5$, then $mincol_{\mathbb{Z}}(L) = 4$.

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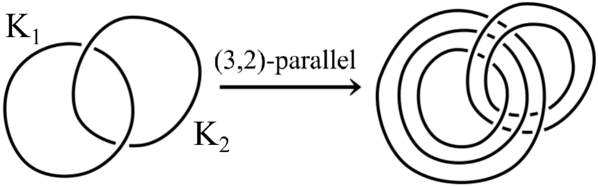
Question

For any $\mathbb{Z}$-colorable link L, $mincol_{\mathbb{Z}}(L) = 4$?

Parallel of a link

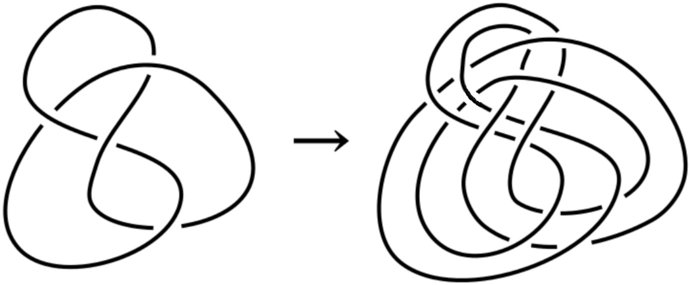
For a link $L = K_1 \cup \cdots \cup K_c$ with a diagram D and a set $(n_1, \cdots, n_c)$ of integers $n_i \geq 1$, we denote by $D^{(n_1,\cdots,n_c)}$ the diagram obtained by taking n_i-parallel copies of the i-th component K_i of D on the plane for $1 \leq i \leq c$.
The link $L^{(n_1,\cdots,n_c)}$ represented by $D^{(n_1,\cdots,n_c)}$ is called the $(n_1, \cdots, n_c)$-parallel of the link L.



When L is a knot, we call (n)-parallel $L^{(n)}$ simply an n-parallel, and denote it by L^n.

Untwisted 2-parallel

A 2-parallel $K^2 = K_1 \cup K_2$ of a knot K is called the untwisted 2-parallel where $\mathrm{lk}(K_1, K_2) = 0$.



Theorem 1

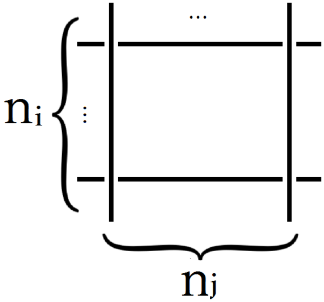
The untwisted 2-parallel K^2 of a knot K is $\mathbb{Z}$-colorable and $mincol_{\mathbb{Z}}(K^2) = 4$.

Theorem 2

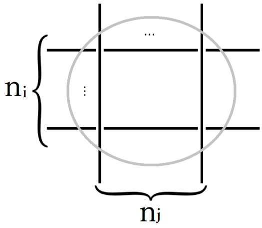
For any diagram of a c-component link L and any even number $n_1, \cdots, n_c$ at least 4, $L^{(n_1,\cdots,n_c)}$ is $\mathbb{Z}$-colorable and $mincol_{\mathbb{Z}}(L^{(n_1,\cdots,n_c)}) = 4$.

Outline of proof of Theorem 2

Let $L = K_1 \cup \cdots \cup K_c$ be a link, and D a diagram of L.
We focus on crossings on $D^{(n_1,\cdots,n_c)}$ obtained by taking parallel copies at a crossing of D.



We put a circle as fencing the crossings.



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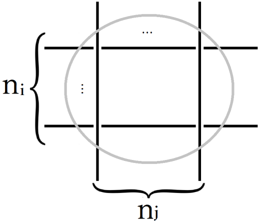
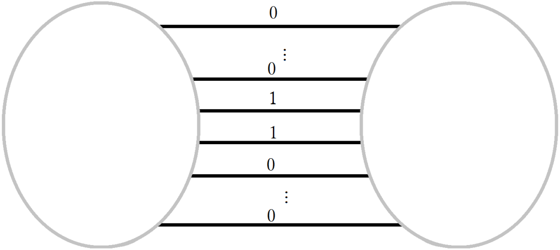
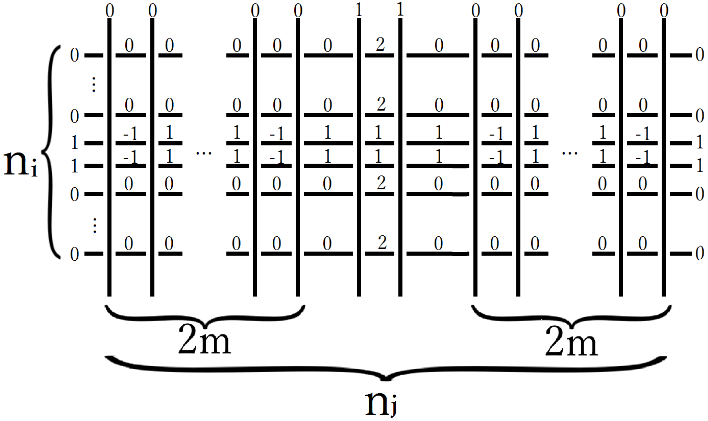


For any parallel arcs $(a_1, \cdots, a_k)$ out of the circle, we fix the colors of $a_{k/2}$ and $a_{k/2+1}$ are 1 and others are 0.

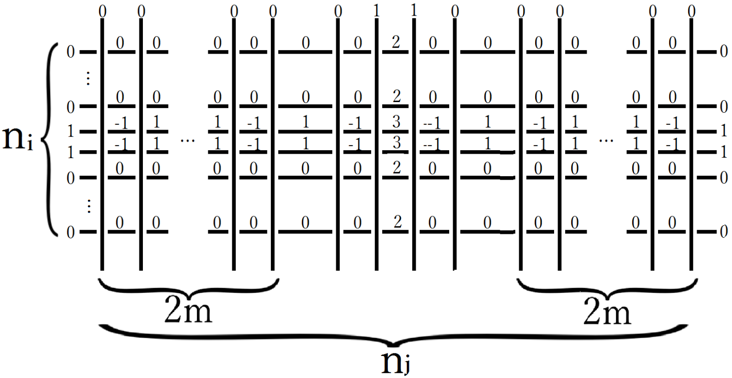


For any arcs inside the circle, we assign colors as follows.

In the case $n_j = 4m + 2 (m \in \mathbb{N})$, we assign the colors $-1, 0, 1, 2$.

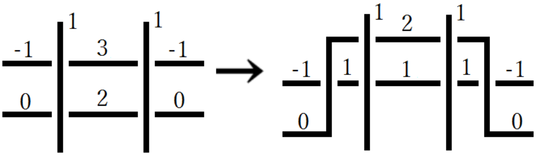


In the case $n_j = 4m + 4 (m \in \mathbb{N})$, we assign the colors $-1, 0, 1, 2, 3$.



We see that $D^{(n_1,\cdots,n_c)}$ admits a $\mathbb{Z}$-coloring C such that $\mathrm{Im}(C) = \{-1, 0, 1, 2, 3\}$. Therefore $L^{(n_1,\cdots,n_c)}$ is $\mathbb{Z}$-colorable.

Moreover, we eliminate the arcs colored by 3 as follows.



It follows $mincol_{\mathbb{Z}}(L^{(n_1,\cdots,n_c)}) = 4$.

□

Thank you for your attention.