Extension of the Interior Polynomial to Signed Bipartite Graphs

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Outline

$G = (V, E)$: a graph

Definition 1 (Tutte polynomial)

$$T_G(x, y) = \sum_{S \subseteq E} (x-1)^{k(S)-k(G)} (y-1)^{k(S)+|S|-|V|},$$

where $k(S)$ denotes the number of connected components of the graph (V, S).

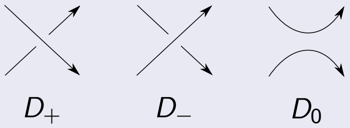
$k(S) + |S| - |V| =$ the first Betti number of the graph (V, S)

Definition 2 (HOMFLY polynomial)

There is a function $P : \{oriented\ links\ in\ S^3\} \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ defined uniquely by

(i) $P(unknot) = 1$

(ii) $v^{-1}P_{D_+} - vP_{D_-} = zP_{D_0}$, where D_+, D_-, D_0 are oriented skein triple.

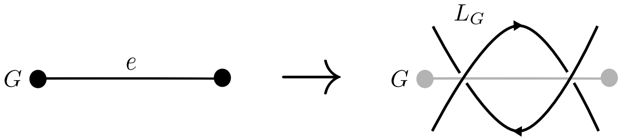


$G = (V, E)$: a connected planar graph

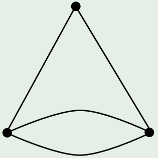
Theorem 3 (F. Jaeger, 1988)

L_G : link obtained from G by replacing each edge by a pair of alternating crossing

$$P_{L_G}(v, z) = \left(\frac{v}{z}\right)^{|V|-1} (vz)^{|E|} T_G \left(\frac{1}{v^2}, \frac{z^2 - v^2 + 1}{z^2}\right).$$



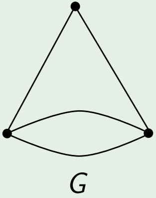
Example



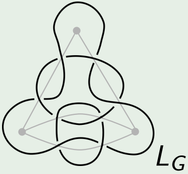
G

$$
\begin{aligned}
T_G(x,y) \quad = \quad & +1x^2 \quad +1x^1 \\
& \qquad\qquad +1x^1y^1 \quad +1y^1 \\
& \qquad\qquad\qquad\qquad +1y^2
\end{aligned}
$$

Example

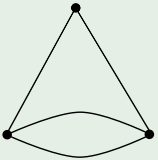


$\longrightarrow$



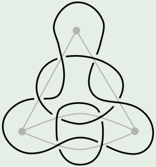
$$T_G(x,y) = +1x^2 \quad +1x^1 \quad +1x^1y^1 \quad +1y^1 \quad +1y^2$$

Example



G

$\longrightarrow$

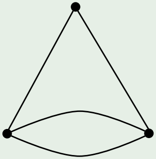


L_G

$$
\begin{array}{lllll}
T_G(x,y) = & +1x^2 & +1x^1 & & \\
& & +1x^1y^1 & +1y^1 & \\
& & & +1y^2 &
\end{array}
$$

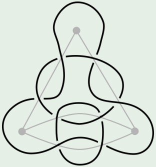
$$
\begin{array}{llllll}
P_{L_G}(v,z) = & +1v^2z^2 & +2v^4z^2 & +2v^6z^2 & & \\
& & +1v^4 & +2v^6 & -3v^8 & \\
& & & +1v^6z^{-2} & -2v^8z^{-2} & +1v^{10}z^{-2}
\end{array}
$$

Example



G

$\longrightarrow$



L_G

$$
\begin{aligned}
T_G(x,y) &= +1x^2 \quad +1x^1 \\
&\qquad\qquad +1x^1y^1 \quad +1y^1 \\
&\qquad\qquad\qquad\qquad +1y^2 \\
T_G(1/x,1) &= +1x^{-2} \quad +2x^{-1} \quad +2 \\
P_{L_G}(v,z) &= +1v^2z^2 \quad +2v^4z^2 \quad +2v^6z^2 \\
&\qquad\qquad +1v^4 \quad +2v^6 \quad -3v^8 \\
&\qquad\qquad\qquad\qquad +1v^6z^{-2} \quad -2v^8z^{-2} \quad +1v^{10}z^{-2}
\end{aligned}
$$

$G = (V, E, \mathcal{E})$: a connected bipartite graph
We order the set E (called hyperedge set).

Definition 4

A hypertree in G is defined to be a function $\mathbf{f} : E \to \mathbb{N} = \{0, 1, 2, \dots\}$ so that a spanning tree of G can be found with valence $\mathbf{f}(e) + 1$ at each $e \in E$.

Q_G : hypertree set of G

Definition 5

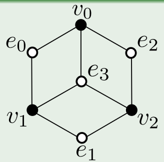
$e \in E$: internally inactive with respect to the hypertree $\mathbf{f}$
$\overset{\text{def}}{\Leftrightarrow}$it is possible to decrease $\mathbf{f}(e)$ by 1 and increase $\mathbf{f}$ at a hyperedge smaller than e by 1 so that another hypertree results.

$\bar{\imath}(\mathbf{f})$ counts internally inactive hyperedges of G with respect to $\mathbf{f}$.

Definition 6 (interior polynomial)

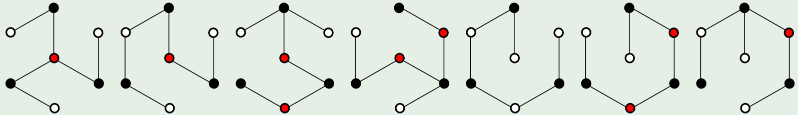
$$I_G(x) = \sum_{\mathbf{f} \in Q_G} x^{\bar{\imath}(\mathbf{f})}.$$

Example



$e_0 \prec e_1 \prec e_2 \prec e_3$

$(0,0,0,2)$ $(1,0,0,1)$ $(0,1,0,1)$ $(0,0,1,1)$ $(1,1,0,0)$ $(0,1,1,0)$ $(1,0,1,0)$

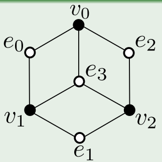


$\bar{\imath}=1$ $\bar{\imath}=1$ $\bar{\imath}=2$ $\bar{\imath}=2$ $\bar{\imath}=0$ $\bar{\imath}=2$ $\bar{\imath}=1$

Definition 6 (interior polynomial)

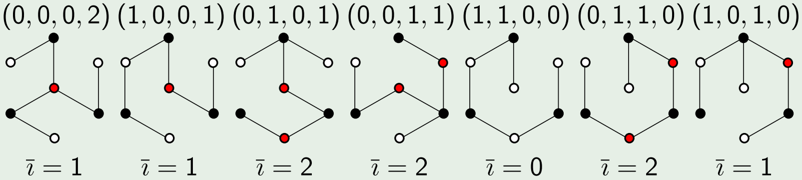
$$I_G(x) = \sum_{\mathbf{f} \in Q_G} x^{\bar{\imath}(\mathbf{f})}.$$

Example



$e_0 \prec e_1 \prec e_2 \prec e_3$

$$I_G(x) = 1x^0 + 3x^1 + 3x^2$$



$G = (V, E, \mathcal{E})$: a connected bipartite graph

Theorem 7 (T. Kálmán, 2013)

$I_G(x)$ *does not depend on the order of hyperedges.*

$\overline{G} = (E, V, \mathcal{E})$: the bipartite graph whose hyperedge set is V

Theorem 8 (T. Kálmán and A. Postnikov, 2016)

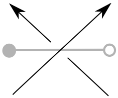
$$I_G = I_{\overline{G}}.$$

Fact

$G = (V, E)$ *: graph*
$\tilde{G}$ *: the bipartite graph obtained from* G *by setting new vertex on each edge*

$$I_{\tilde{G}}(x) = x^{|V|-1} T_G(1/x, 1).$$

For any plane bipartite graph G, L_G is the alternating link obtained from G by replacing each edge by a crossing.



$G = (V, E, \mathcal{E})$: a connected plane bipartite graph

Theorem 9 (T. Kálmán, A. Postnikov and H. Murakami, 2016)

$T_G(v) =$ the coefficient of $z^{|\mathcal{E}|-(|V|+|E|)+1}$ in the HOMFLY polynomial of L_G (top of the HOMFLY polynomial)
Then,

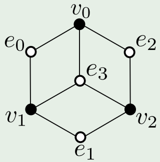
$$T_G(v) = v^{|\mathcal{E}|-(|V|+|E|)+1} I_G(v^2),$$

where $I_G(x)$ is the interior polynomial of G.

Example

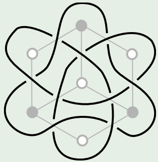
G : bipartite graph

L_G : special alternating link obtained from G



G

$\longrightarrow$



L_G

$$I_G(x) = +1x^0 \quad +3x^1 \quad +3x^2$$

$$\begin{aligned} P_{L_G}(v,z) = &+1v^3z^3 \quad +3v^5z^3 \quad +3v^7z^3 \\ &+3v^5z \quad +4v^7z \quad -4v^9z \\ &+2v^7z^{-1} \quad -3v^9z^{-1} \quad +1v^{11}z^{-1} \end{aligned}$$

G: a (disconnected) bipartite graph
$k(G) =$ the number of components of G
$G = G_1 \cup G_2 \cup \cdots \cup G_{k(G)}$

Definition 10

$$I'_G(x) = (1-x)^{k(G)-1} \prod_{i=1}^{k(G)} I_{G_i}(x).$$

$G \cup G'$: the disjoint union of G and G'

Lemma 11

$$I'_{G \cup G'} = (1-x) I'_G I'_{G'}.$$

Remark

$$P_{L \cup L'}(v, z) = \frac{v^{-1} - v}{z} P_L(v, z) P_{L'}(v, z)$$

Theorem 12

For any plane bipartite graph $G = (V, E, \mathcal{E})$, the top of the HOMFLY polynomial $P_{L_G}(v, z)$ is equal to

$$v^{|\mathcal{E}| - (|V| + |E|) + 1} I'_G \left(v^2\right).$$

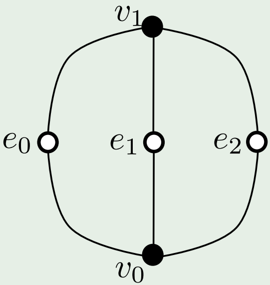
$G = (V, E, \mathcal{E})$: a bipartite graph

Definition 13

For $v \in V$ and $e \in E$, let $\mathbf{v}$ and $\mathbf{e}$ denote the standard generators of $\mathbb{R}^V \oplus \mathbb{R}^E$. Then the root polytope of G is defined to be

$$Q_G = \mathrm{Conv}\{\mathbf{v} + \mathbf{e} \mid ve \text{ is an edge of } G\}.$$

Example



$G = (V, E, \mathcal{E})$: a bipartite graph

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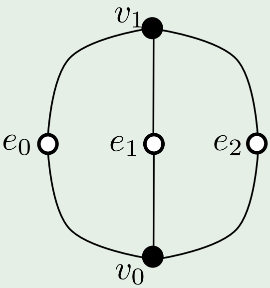

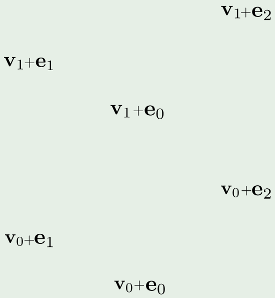

$G = (V, E, \mathcal{E})$: a bipartite graph

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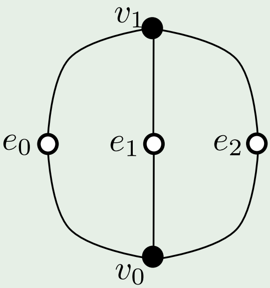
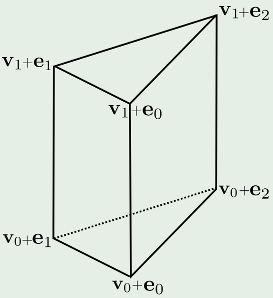
Example





$G = (V, E)$: a connected bipartite graph

Definition 14 (Ehrhart polynoial)

Q_G : a root polytope of a bipartite graph G

$$\varepsilon_{Q_G}(s) := |s \cdot Q_G \cap \mathbb{Z}^V \oplus \mathbb{Z}^E| = \sum_{k=0}^{|V|+|E|-2} a_k \binom{s + |V| + |E| - 2 - k}{|V| + |E| - 2}.$$

Theorem 15 (T. Kálmán and A. Postnikov, 2016)

$\varepsilon_{Q_G}(s) = \sum_{k=0}^{|V|+|E|-2} a_k \binom{s+|V|+|E|-2-k}{|V|+|E|-2}$: the Ehrhart polynomial
Then,

$$I_G(x) = a_0 x^0 + a_1 x^1 + \cdots + a_{|V|+|E|-2} x^{|V|+|E|-2}.$$

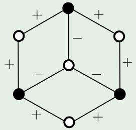
$G = (V, E, \mathcal{E}_- \cup \mathcal{E}_+)$: a connected signed bipartite graph

Definition 16 (signed interior polynomial)

$$I_G^+(x) = \sum_{\mathcal{S} \subseteq \mathcal{E}_-} (-1)^{|\mathcal{S}|} I'_{G \setminus \mathcal{S}}(x),$$

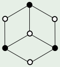
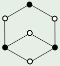
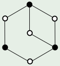
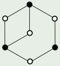
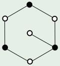
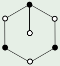
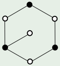
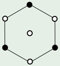
where $G \setminus \mathcal{S}$ is bipartite graph obtained from G by deleting $\forall e \in \mathcal{S}$ and by forgetting sign.

Example



$$I_G^+ = 1x^3$$

Example

		$(-1)^{\lvert\mathcal{S}\rvert} I'_{G\setminus\mathcal{S}}$	
		$1x^0 + 3x^1 + 3x^2$	$\times 1$
  	$-$	$1x^0 - 2x^1 - 2x^2$	$\times 3$
  		$1x^0 + 1x^1 + 1x^2$	$\times 3$
	$-$	$1x^0 + 0x^1 + 0x^2 + 1x^3$	$\times 1$
$I^+_G(x)$	$=$	$0x^0 + 0x^1 + 0x^2 + 1x^3$	

For any signed plane bipartite graph G, L_G is the oriented link obtained from G by replacing each edge to a crossing.

positive edge   negative edge

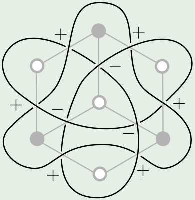
$G = (V, E, \mathcal{E}_+ \cup \mathcal{E}_-)$: plane signed bipartite graph

Theorem 17 (K.)

$T_G(v)$ = the coefficient of $z^{|\mathcal{E}_+|+|\mathcal{E}_-|-(|V|+|E|)+1}$ in the HOMFLY polynomial of L_G Then,

$$T_G(v) = v^{|\mathcal{E}_+|-|\mathcal{E}_-|-(|V|+|E|)+1} I_G^+\left(v^2\right).$$

Example



$$I_G^+ = 1x^3$$

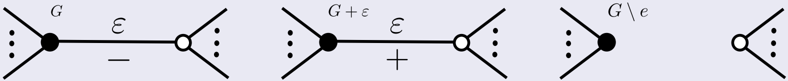
$$P_{L_G}(v,z) = +1v^3z^3 +4v^3z -1v^5z -1vz^{-1} +3v^3z^{-1} -2v^5z^{-1}$$

Proof of Thm. 17.

We prove by induction on $|\mathcal{E}_-|$.

base step : Thm. 12

inductive step : When $|\mathcal{E}_-| < m$, suppose that this theorem holds. Let e be a negative edge.



$$
\begin{aligned}
& I^+_G(v^2) \\
= & \sum_{\substack{\mathcal{S} \subseteq \mathcal{E}_-(G) \\ e \notin \mathcal{S}}} (-1)^{|\mathcal{S}|} I'_{G \setminus \mathcal{S}}(v^2) + \sum_{\substack{\mathcal{S} \subseteq \mathcal{E}_-(G) \\ e \in \mathcal{S}}} (-1)^{|\mathcal{S}|} I'_{G \setminus \mathcal{S}}(v^2) \\
= & I^+_{G+e}(v^2) - I^+_{G \setminus e}(v^2) \\
= & \frac{1}{v^{|\mathcal{E}_+| - |\mathcal{E}_-| - (|V| + |E|) + 1}} \left(v^{-2} \mathrm{Top}(P_{L_{G+e}})(v) - v^{-1} \mathrm{Top}(P_{L_{G \setminus e}})(v) \right) \\
= & \frac{1}{v^{|\mathcal{E}_+| - |\mathcal{E}_-| - (|V| + |E|) + 1}} \mathrm{Top}(P_{L_G})(v).
\end{aligned}
$$

Good point

Any oriented link has a special diagram (constructed from a signed bipartite graph).

Bad point

There exist bipartite graphs G such that $I_G^+(x) = 0$.

Theorem 18 (Morton,1986)

For any diagram D of L,

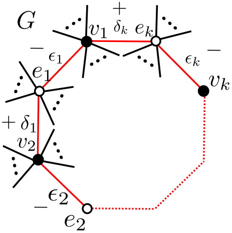
$$maxdeg_z\{P_L(v,z)\} \leq c(D) - s(D) + 1,$$

where $c(D)$ is the number of crossing of D and $s(D)$ is the number of the Seifert circles of D.

If D is obtained from bipartite graph $G = (V, E, \mathcal{E})$, $c(D) - s(D) + 1 = |\mathcal{E}| - (|V| + |E|) + 1.$

Theorem 19 (K.)

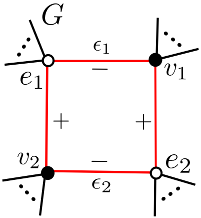
If G contains contains an alternate cycle of positive and negative edges, then $I_G^+ = 0$.



Corollary 20

If D is represented by G which contains an alternate cycle, then Morton's inequality is not sharp.

Proof of Thm. 19



G : a bipartite graph containing an alternate cycle of length k

$k = 2$

First, We want to show that $Q_G = Q_{G\setminus\epsilon_1} \cup Q_{G\setminus\epsilon_2}$.

Clearly, $Q_G \supset Q_{G\setminus\epsilon_1} \cup Q_{G\setminus\epsilon_1}$.

$a_1 := \mathbf{e}_1 + \mathbf{v}_1, b_1 := \mathbf{e}_1 + \mathbf{v}_2, a_2 := \mathbf{e}_2 + \mathbf{v}_2, b_2 := \mathbf{e}_2 + \mathbf{v}_1$.

$$x = \lambda_1 a_1 + \lambda_2 b_1 + \lambda_3 a_2 + \lambda_4 b_2 + \cdots \in Q_G.$$

Case $\lambda_1 \leq \lambda_3$.

$$x = (\lambda_1 - \lambda_1)a_1 + (\lambda_2 + \lambda_1)b_1 + (\lambda_3 - \lambda_1)a_2 + (\lambda_4 + \lambda_1)b_2 + \cdots \in Q_{G\setminus\epsilon_1}.$$

Case $\lambda_1 \geq \lambda_3$.

$$x = (\lambda_1 - \lambda_3)a_1 + (\lambda_2 + \lambda_3)b_1 + (\lambda_3 - \lambda_3)a_2 + (\lambda_4 + \lambda_3)b_2 + \cdots \in Q_{G\setminus\epsilon_2}.$$

So, We show that $Q_G \subset Q_{G\setminus\epsilon_1} \cup Q_{G\setminus\epsilon_1}$.
Now,

$$|s \cdot (Q_{G\setminus\epsilon_1} \cup Q_{G\setminus\epsilon_2}) \cap \mathbb{Z}^n| - |s \cdot Q_{G\setminus e_0} \cap \mathbb{Z}^n| \\ - |s \cdot Q_{G\setminus e_1} \cap \mathbb{Z}^n| + |s \cdot (Q_{G\setminus\epsilon_1} \cap Q_{G\setminus\epsilon_2}) \cap \mathbb{Z}^n| = 0.$$

By using some properties, we show that

$$|s \cdot (Q_{G\setminus\epsilon_1} \cap Q_{G\setminus\epsilon_2}) \cap \mathbb{Z}^n| = |s \cdot (Q_{G\setminus\epsilon_1\setminus\epsilon_2}) \cap \mathbb{Z}^n|.$$

Therefore,

$$\begin{array}{ccccccccc} \varepsilon_{Q_G} & - & \varepsilon_{Q_{G\setminus e_0}} & - & \varepsilon_{Q_{G\setminus e_1}} & + & \varepsilon_{Q_{G\setminus e_0\setminus e_1}} & = 0 \\ I_G & - & I_{G\setminus\epsilon_1} & - & I_{G\setminus\epsilon_2} & + & I_{G\setminus\epsilon_1\setminus\epsilon_2} & = 0. \end{array}$$

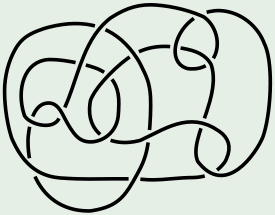
Question

Morton's inequality is not sharp $\Rightarrow$ G contains an alternate cycle?

Answer : No

Example (M. Brittenham and J. Jensen, 2006)

K : 15_{100154}



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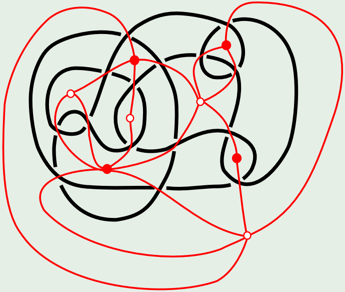

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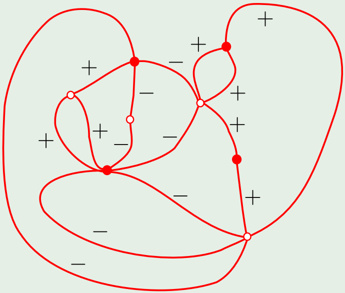

$|\mathcal{E}| - (|V| + |E|) + 1 = 15 - 8 + 1 = 8$, $maxdeg_z\{P_K(v,z)\} = 6$.