

# Forbidden detour move and Jones polynomials

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結び目の数学 IX

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Forbidden detour  
move and Jones  
polynomials.

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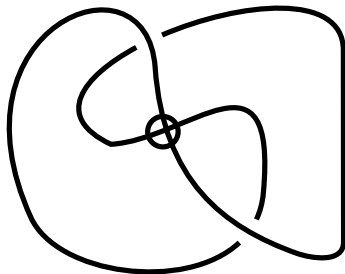
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Outline of proof

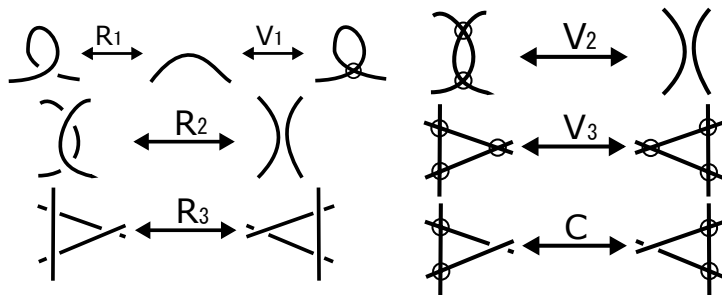
## Definition (virtual knot)

A *virtual knot* is an equivalence class of virtual knot diagrams under Reidemeister moves and virtual Reidemeister moves.

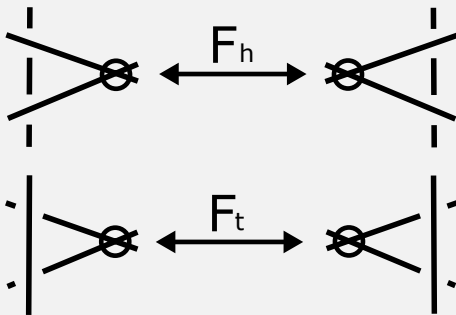


## Definition (virtual knot)

A *virtual knot* is an equivalence class of virtual knot diagrams under Reidemeister moves and virtual Reidemeister moves.



## Definition (forbidden move)



# Why "forbidden" ? (1)

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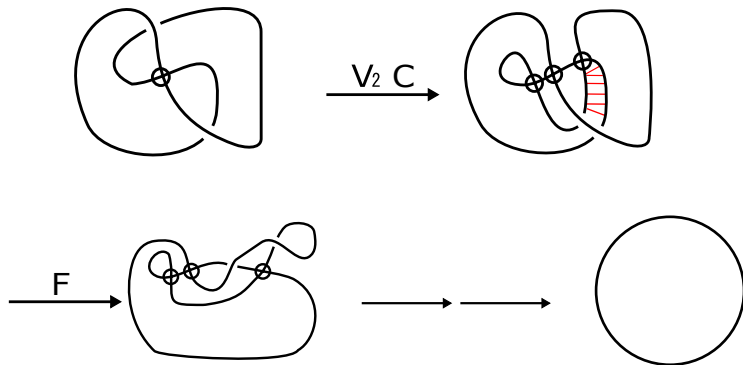
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Theorem [Kanenobu, 2001][Nelson, 2001]

For any diagram  $D$  of a virtual knot  $K$ , there exists a finite sequence of Reidemeister moves, virtual Reidemeister moves and forbidden moves that takes  $D$  to a trivial knot diagram.



# Forbidden detour move

Forbidden detour  
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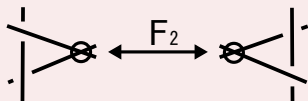
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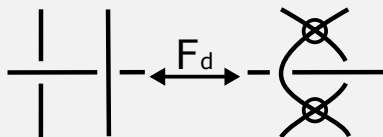
Outline of proof

Theorem [Kanenobu, 2001][Nelson, 2001]

An  $F_2$ -move can be realized by twice forbidden moves and some classical and virtual Reidemeister moves.



Definition (Forbidden detour move)[c.f. Crans-Meller-Ganzell]



Remark  $F_d$ -move can be realized by a  $V_2$ -move and an  $F_2$ -move.

# Why "forbidden" ? (2)

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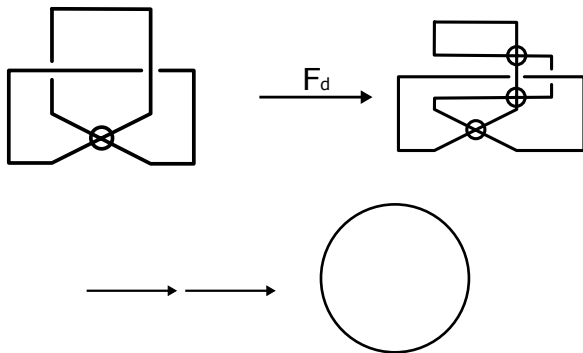
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## Theorem [Y.]

For any diagram  $D$  of a virtual knot  $K$ , there exists a finite sequence of Reidemeister moves, virtual Reidemeister moves and **forbidden detour moves** that takes  $D$  to a trivial knot diagram.



# Bracket polynomial and Jones polynomial

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## Definition (Bracket polynomial)

Rule 1:  $\langle \bigcirc \rangle = 1$

Rule 2:  $\langle \text{crossing} \rangle = A \langle \text{right curl} \rangle \langle \text{left curl} \rangle + A^{-1} \langle \text{right curl} \rangle \langle \text{left curl} \rangle$

$\langle \text{crossing} \rangle = A \langle \text{right curl} \rangle \langle \text{left curl} \rangle + A^{-1} \langle \text{right curl} \rangle \langle \text{left curl} \rangle$

Rule 3:  $\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle = d \langle D \rangle$

## Definition (Jones polynomial)

Let  $K$  be a virtual knot and  $w$  a writhe of a diagram  $D$  of  $K$ .

$$f_K(A) = (-A^3)^{-w} \langle D \rangle$$

Remark:  $V_K(t) = (-t^{-3/4})^{-w(D)} \langle D \rangle$ ,  $A = t^{-1/4}$



## Theorem [Ganzell, 2014]

Let  $D$  and  $D'$  be diagrams of virtual knots  $K$  and  $K'$ , respectively. If  $D$  and  $D'$  can be transformed into each other by a single forbidden move, then  $f_K(A) - f_{K'}(A)$  is divisible by  $A^{10} - A^6 - A^4 + 1$ .

## Theorem [Y.]

Let  $D$  and  $D'$  be diagrams of virtual knots  $K$  and  $K'$ , respectively. If  $D$  and  $D'$  can be transformed into each other by a **single forbidden detour move**, then  $f_K(A) - f_{K'}(A)$  is divisible by  $A^{10} - A^6 - A^4 + 1$ .

Remark: The divisors of two theorems are same, but quotients are different.

# Outline of proof (1)

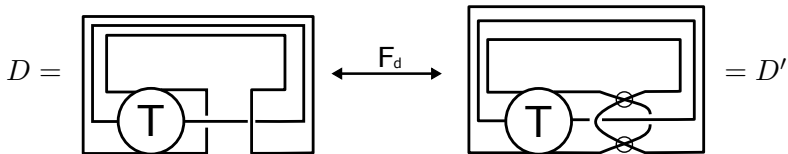
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$$\langle \text{Diagram with T} \rangle = \sum_{i=1}^{15} p_i \langle \text{Diagram with F}_i \rangle$$

$$\langle \text{Diagram with T and loop} \rangle = \sum_{i=1}^{15} p_i \langle \text{Diagram with F}_i \text{ and loop} \rangle$$

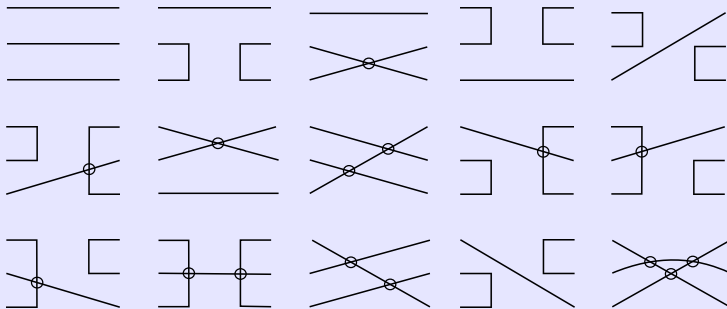
# Outline of proof (2)

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Lemma [cf. Ganzell, 2014]

$$\langle \text{T} \rangle = \sum_{i=1}^{15} p_i \langle F_i \rangle$$



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# Outline of proof (3)

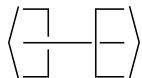
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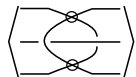
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$$= A^2 \langle \text{Diagram } G_1 \rangle + \langle \text{Diagram } G_2 \rangle + \langle \text{Diagram } G_3 \rangle + A^{-2} \langle \text{Diagram } G_4 \rangle$$

$G_1 \qquad G_2 \qquad G_3 \qquad G_4$



$$= A^2 \langle \text{Diagram } G'_1 \rangle + \langle \text{Diagram } G'_2 \rangle + \langle \text{Diagram } G'_3 \rangle + A^{-2} \langle \text{Diagram } G'_4 \rangle$$

$G'_1 \qquad G'_2 \qquad G'_3 \qquad G'_4$

# Outline of proof (4)

$$\langle \text{T} \rangle = A^2 \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle + \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle$$

$$+ \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle + A^{-2} \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle$$

$$\langle \text{T} \rangle = A^2 \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle + \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle$$

$$+ \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle + A^{-2} \sum_{i=1}^{15} p_i \langle \text{F}_i \rangle$$

# Outline of proof (5)

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$$\alpha_{i,j} = \text{Diagram of } F_i \text{ and } G_j \text{ on two strands} \\ = \langle \widehat{\alpha_{i,j}} \rangle$$

The diagram shows two horizontal strands with two circles labeled  $F_i$  and  $G_j$  between them. The second diagram shows the same two circles, but they are enclosed within a rectangular frame that is itself inside a larger, double-lined frame.

$$\beta_{i,j} = \text{Diagram of } F_i \text{ and } G'_j \text{ on two strands} \\ = \langle \widehat{\beta_{i,j}} \rangle$$

The diagram shows two horizontal strands with two circles labeled  $F_i$  and  $G'_j$  between them. The second diagram shows the same two circles, but they are enclosed within a rectangular frame that is itself inside a larger, double-lined frame.

# Outline of proof (6)

$$\langle \widehat{\alpha}_{8,1} \rangle = \langle \text{Diagram} \rangle$$

$$= \langle \bigcirc \rangle = 1$$

$$\langle \widehat{\beta}_{8,1} \rangle = \langle \text{Diagram} \rangle$$

$$= \langle \bigcirc \bigcirc \rangle = d$$



# Outline of proof (7)

$$\begin{aligned}\langle D \rangle = & p_1(A^2d + 1 + 1 + A^{-2}d) & \langle D' \rangle = & p_1(A^2d + 1 + 1 + A^{-2}d) \\ & + p_2(A^2 + d + d + A^{-2}d^2) & & + p_2(A^2d + 1 + 1 + A^{-2}) \\ & + p_3(A^2 + 1 + 1 + A^{-2}d) & & + p_3(A^2d^2 + d + d + A^{-2}) \\ & + p_4(A^2d^2 + d + d + A^{-2}) & & + p_4(A^2 + 1 + 1 + A^{-2}d) \\ & + p_5(A^2d + 1 + d^2 + A^{-2}d) & & + p_5(A^2 + 1 + d + A^{-2}) \\ & + p_6(A^2d + 1 + d + A^{-2}) & & + p_6(A^2d + d + 1 + A^{-2}) \\ & + p_7(A^2d + 1 + 1 + A^{-2}) & & + p_7(A^2d + d + d + A^{-2}d^2) \\ & + p_8(A^2 + 1 + d + A^{-2}) & & + p_8(A^2d + 1 + d^2 + A^{-2}d) \\ & + p_9(A^2 + d + 1 + A^{-2}d) & & + p_9(A^2 + 1 + d + A^{-2}d) \\ & + p_{10}(A^2 + 1 + d + A^{-2}d) & & + p_{10}(A^2 + d + 1 + A^{-2}d) \\ & + p_{11}(A^2d + d + 1 + A^{-2}) & & + p_{11}(A^2d + 1 + d + A^{-2}) \\ & + p_{12}(A^2 + 1 + 1 + A^{-2}) & & + p_{12}(A^2 + 1 + 1 + A^{-2}) \\ & + p_{13}(A^2 + d + 1 + A^{-2}) & & + p_{13}(A^2d + d^2 + 1 + A^{-2}d) \\ & + p_{14}(A^2d + d^2 + 1 + A^{-2}d) & & + p_{14}(A^2 + d + 1 + A^{-2}) \\ & + p_{15}(A^2 + d + d + A^{-2}) & & + p_{15}(A^2 + d + d + A^{-2})\end{aligned}$$

# Outline of proof (7)

$$\begin{aligned}\langle D \rangle = & p_1(A^2d + 1 + 1 + A^{-2}d) & \langle D' \rangle = & p_1(A^2d + 1 + 1 + A^{-2}d) \\ & + p_2(A^2 + d + d + A^{-2}d^2) & & + p_2(A^2d + 1 + 1 + A^{-2}) \\ & + p_3(A^2 + 1 + 1 + A^{-2}d) & & + p_3(A^2d^2 + d + d + A^{-2}) \\ & + p_4(A^2d^2 + d + d + A^{-2}) & & + p_4(A^2 + 1 + 1 + A^{-2}d) \\ & + p_5(A^2d + 1 + d^2 + A^{-2}d) & & + p_5(A^2 + 1 + d + A^{-2}) \\ & + p_6(A^2d + 1 + d + A^{-2}) & & + p_6(A^2d + d + 1 + A^{-2}) \\ & + p_7(A^2d + 1 + 1 + A^{-2}) & & + p_7(A^2d + d + d + A^{-2}d^2) \\ & + p_8(A^2 + 1 + d + A^{-2}) & & + p_8(A^2d + 1 + d^2 + A^{-2}d) \\ & + p_9(A^2 + d + 1 + A^{-2}d) & & + p_9(A^2 + 1 + d + A^{-2}d) \\ & + p_{10}(A^2 + 1 + d + A^{-2}d) & & + p_{10}(A^2 + d + 1 + A^{-2}d) \\ & + p_{11}(A^2d + d + 1 + A^{-2}) & & + p_{11}(A^2d + 1 + d + A^{-2}) \\ & + p_{12}(A^2 + 1 + 1 + A^{-2}) & & + p_{12}(A^2 + 1 + 1 + A^{-2}) \\ & + p_{13}(A^2 + d + 1 + A^{-2}) & & + p_{13}(A^2d + d^2 + 1 + A^{-2}d) \\ & + p_{14}(A^2d + d^2 + 1 + A^{-2}d) & & + p_{14}(A^2 + d + 1 + A^{-2}) \\ & + p_{15}(A^2 + d + d + A^{-2}) & & + p_{15}(A^2 + d + d + A^{-2})\end{aligned}$$

# Outline of proof (8)

Since the writhe of  $D$  equals the writhe of  $D'$ , we have

$$\begin{aligned} & f_K(A) - f_{K'}(A) \\ &= (-A^3)^{-w} \{ p_2(A^4 - 1 - A^{-2} + A^{-6}) \\ &\quad + p_3(-A^6 + A^2 + 1 - A^{-4}) \\ &\quad + p_4(A^6 - A^2 - 1 + A^{-4}) \\ &\quad + p_7(-A^4 + 1 + A^{-2} - A^{-6}) \} \\ &= (-A^3)^{-w} (p_2A^{-6} - p_3A^{-4} + p_4A^{-4} - p_7A^{-6})(A^{10} - A^6 - A^4 + 1) \end{aligned}$$

□