On the colored $\mathfrak{sl}_3$ Jones polynomial for 2-bridge links

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Quantum Integers

Let n and k be non-negative integers such that $k \leq n$. We define a quantum integer by

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

and a version of q-binomial coefficient by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!},$$

where $[n]! = \prod_{i=1}^{n}[i]$.

A *q-pochhammer symbol* is defined as

$$(q;q)_n = \prod_{i=1}^{n}(1-q^i).$$

We define another version of *q-binomial coefficient* by

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$

and a *q-multinomial coefficient* by

$$\binom{n}{n_1, n_2, \ldots, n_m}_q = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n_2}\ldots(q;q)_{n_m}},$$

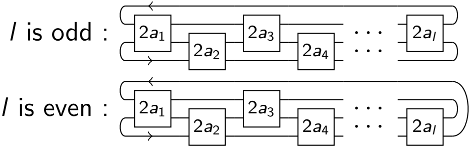
where m and $n_1, n_2, \ldots, n_m$ are non-negative integers such that $n_1 + n_2 + \cdots + n_m = n$.

Remark

$\begin{bmatrix} n \\ k \end{bmatrix} = q^{\frac{-n(n-k)}{2}} \binom{n}{k}_q$

2-bridge link diagrams

The 2-bridge links are represented by the following link diagrams.



where $a_1, a_2, \ldots, a_l$ be non-zero integers and the boxed $2a_i$ represents the right-handed (resp. left-handed) $|a_i|$ full twists if $a_i > 0$ (resp. $a_i < 0$). We denote this link diagram by $[2a_1, 2a_2, \ldots, 2a_l]$.

The colored $\mathfrak{sl}_3$ Jones polynomials of type (n,0) for 2-bridge links

Theorem 1 (From full twists formulas [Y.])

$$J^{\mathfrak{sl}_3}_{(n,0)}([2a_1, 2a_2, \ldots, 2a_l]\,;q)$$
$$= \prod_{j=0}^{l-1} \sum_{0 \le k^{(j+1)}_{|a_{j+1}|} \le \cdots \le k^{(j+1)}_1 \le K_j} q^{\varepsilon_{j+1}(K_j - k^{(j+1)}_{|a_{j+1}|})} q^{\varepsilon_{j+1} \sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)2} + 2k_i^{(j+1)})}$$
$$\times \frac{(q^{\varepsilon_{j+1}})_{K_j}}{(q^{\varepsilon_{j+1}})_{k^{(j+1)}_{|a_{j+1}|}}} \binom{n}{k_1^{(j+1)\prime}, k_2^{(j+1)\prime}, \ldots, k^{(j+1)\prime}_{|a_{j+1}|}, k^{(j+1)}_{|a_{j+1}|}}_{q^{\varepsilon_{j+1}}}$$
$$\times q^{-(n-K_l)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{K_l+1})(1-q^{K_l+2})},$$

where $\varepsilon_{j+1} = \frac{a_{j+1}}{|a_{j+1}|}$, $K_0 = n, K_j = n - k^{(j)}_{|a_j|}$ *and* $k^{(j)}_0 = K_j, k^{(j+1)\prime}_{|a_{i+1}|} = k^{(j)}_i - k^{(j)}_{i+1}$.

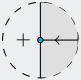
Theorem 2 (From colored trivalent graphs [Y.])

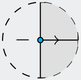
$$
\begin{aligned}
&J^{\mathfrak{sl}_3}_{(n,0)}([2a_1, 2a_2, \ldots, 2a_l]\,; q) \\
&\quad = \sum_{0 \leq i_1, i_2 \ldots i_l \leq n} \frac{\Delta(i_1, i_1)}{\Delta(n, 0)} \frac{\theta(n, n, (i_l, i_l))}{\theta(n, n, (i_1, i_1))} q^{-\frac{2}{3}(n^2+3n)(\sum_{k=1}^{l} a_k)} q^{\sum_{k=1}^{l} a_k (i_k^2 + 2i_k)} \\
&\qquad \times \prod_{k=1}^{l-1} \begin{Bmatrix} n & n & (i_{k+1}, i_{k+1}) \\ n & n & (i_k, i_k) \end{Bmatrix}.
\end{aligned}
$$

A_2 web spaces

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$ be a m-tuple of signs $+$ or $-$. Let D_ε denote D_m whose marked point $\exp(2\pi\sqrt{-1}/m)^{j-1}$ is decorated by ε_j for $j = 1, 2, \dots, m$.

Definition 3

- A bipartite uni-trivalent graph G is a directed graph such that every vertex is either trivalent or univalent, and all edges adjacenting to a trivalent vertex point toward it or point away from it.
- A bipartite trivalent graph on D_ε is an embedded bipartite uni-trivalent graph into D_m such that each univalent vertices are either 

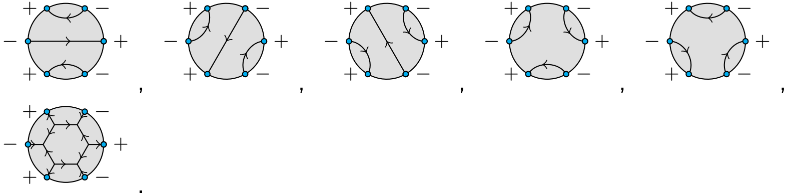
or 

.

Definition 4

An A_2 basis web in D_ε is the boundary-fixing isotopy class of a bipartite trivalent graph G in D_ε, where any internal face of $D \setminus G$ has at least six sides. Let B_ε be the set of all A_2 basis webs in D_ε. The A_2 web space W_ε is the $\mathbb{Q}(q^{\frac{1}{6}})$-vector space spanned by B_ε,

For example, $B_{(+,-,+,-,+,-)}$ has the following A_2 basis webs:

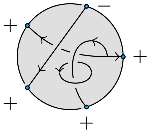


Definition 5

- A tangled trivalent graph diagram G in D_ε is an immersed bipartite uni-trivalent graph in D_ε whose intersection points are only transverse double points of edges with crossing data  or  .
- A tangled trivalent graph in D_ε is the regular isotopy class of a tangled trivalent graph.

c.f.  .

The right diagram is an example of a tangled trivalent graph diagram in $D_{(+,-,+,+,+)}$.

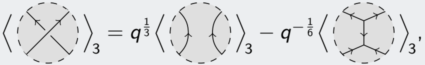


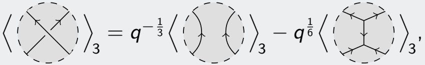
The A_2 bracket

Let us denote the set of all tangled trivalent graphs in D_ε by T_ε.

Definition 6 (Kuperberg)

We define a $\mathbb{Q}(q^{\frac{1}{6}})$-linear map $\langle \cdot \rangle_3 \colon \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \to W_\varepsilon$ by the following.

- 

- 

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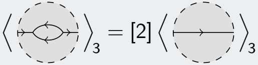
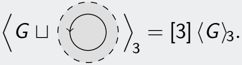
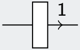
- 

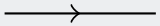
- 

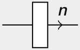
The A_2 clasps

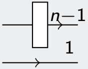
Definition 7 (the A_2 clasp of type $(n, 0)$)

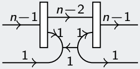
We define recursively A_2 clasps by the following.

- 

$=$  $\in W_{1^+ + 1^-}$

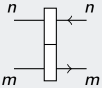
- 

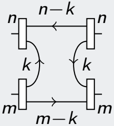
$= \Big\langle$ 

$\Big\rangle_3 - \dfrac{[n-1]}{[n]} \Big\langle$ 

$\Big\rangle_3 \in W_{n^+ + n^-}$

Definition 8 (the A_2 clasp of type (n, m))

$\Big\langle$ 

$\Big\rangle_3 = \displaystyle\sum_{k=0}^{\min\{m,n\}} (-1)^k \frac{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix}}{\begin{bmatrix} n+m+1 \\ k \end{bmatrix}} \Big\langle$ 

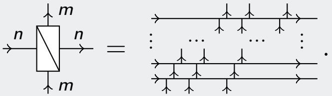
$\Big\rangle_3$

Colored vertices

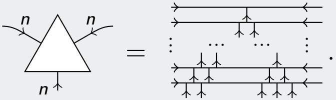
Let m, n be non-negative integers.

Definition 9 (Y.)

A colored 4-valent vertex is defined by



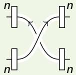
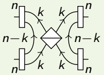
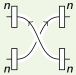
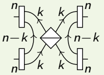
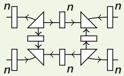
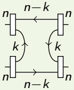
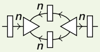
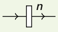
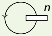
A colored trivalent vertex is defined by



Colored A_2 bracket skein relations

Let n be a positive integer.

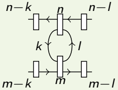
Theorem 10 (Y.)

1. $\big\langle$  $\big\rangle_3 = \sum_{k=0}^{n}(-1)^k q^{\frac{2n^2-6nk+3k^2}{6}}\binom{n}{k}_q \big\langle$  $\big\rangle_3$
2. $\big\langle$  $\big\rangle_3 = \sum_{k=0}^{n}(-1)^k q^{\frac{-2n^2+3k^2}{6}}\binom{n}{k}_q \big\langle$  $\big\rangle_3$
3. $\big\langle$  $\big\rangle_3 = \sum_{k=0}^{n} \big\langle$  $\big\rangle_3$
4. $\big\langle$  $\big\rangle_3 = [n+1]\big\langle$  $\big\rangle_3$
5. $\big\langle$  $\big\rangle_3 = \frac{[n+1]\,[n+2]}{[2]}\emptyset$

The A_2 bracket bubble skein expansion formula

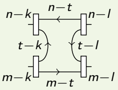
Let $m, n \geq k, l$ be positive integers.

Theorem 11 (Y.)

$\Big\langle$ 

$\Big\rangle_3$

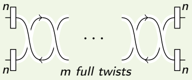
$$= \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} \frac{\begin{bmatrix} n \\ t \end{bmatrix} \begin{bmatrix} m \\ t \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix} \begin{bmatrix} t \\ l \end{bmatrix} \begin{bmatrix} n+m-t+2 \\ n+m-k-l+2 \end{bmatrix}}{\begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} \begin{bmatrix} m \\ l \end{bmatrix}}$$

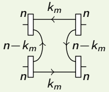
$\Big\langle$ 

$\Big\rangle_3$

The m full twists formula

Theorem 12 (Y.)

$\Big\langle$  $\Big\rangle_3 = q^{-\frac{2m}{3}(n^2+3n)} \sum_{0 \le k_m \le \cdots \le k_1 \le n} (-1)^{n-k_m} q^{n-k_m} q^{\sum_{i=1}^{m}(k_i^2+2k_i)}$

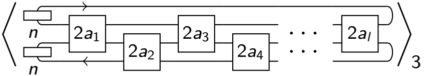
$\times \dfrac{(q)_n}{(q)_{k_m}} \dbinom{n}{k'_1, k'_2, \ldots, k'_m, k_m}_q$ $\Big\langle$  $\Big\rangle_3$,

where k_i, k'_i are integers such that $k_0 = n$, $k'_{i+1} = k_i - k_{i+1}$ for $i = 0, 1, \ldots, m-1$.

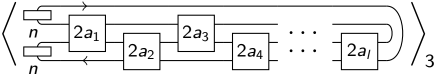
Definition of the $\mathfrak{sl}_3$ colored Jones polynomial for 2-bridge links

$J^{\mathfrak{sl}_3}_{(n,0)}([2a_1, 2a_2, \ldots, 2a_l]\,; q)$

$=$

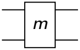


$/\Delta(n,0)$ if l is odd,

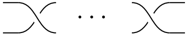


$/\Delta(n,0)$ if l is even,

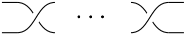
where, $a_1, a_2, \ldots, a_l$ are non-zero integers and



$=$

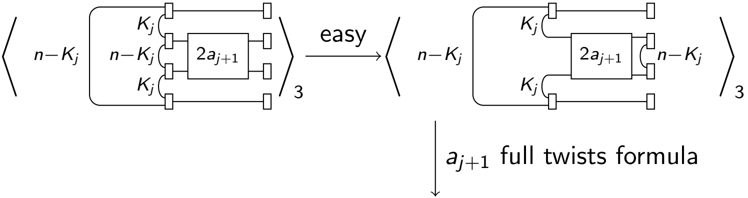


right-handed m half twists if $m > 0$,

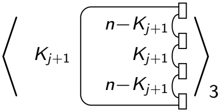


left-handed m half twists if $m < 0$.

Proof of Theorem 1

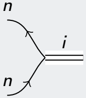


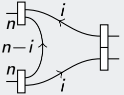
$$K_{j+1} = n - k^{(j+1)}_{|a_{j+1}|}$$



colored trivalent graphs

Definition 13

Let n be a non-negative integer. For $0 \leq i \leq n$, we define a colored trivalent graph 

by 

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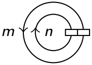
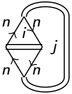
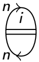
We use the following notations:

- $\Delta(m, n) = \Big\langle$ 

$\Big\rangle_3$, $\mathrm{Tet}\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix} = \Big\langle$ 

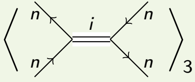
$\Big\rangle_3$,
- $\theta(n, n, (i, i)) = \Big\langle$ 

$\Big\rangle_3$, $\begin{Bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{Bmatrix} = \dfrac{\mathrm{Tet}\begin{bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{bmatrix} \Delta(j,j)}{\theta(n, n, (j, j))^2}$,

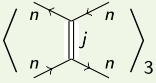
where m, n are any non-negative integers and $0 \leq i, j \leq n$.

Recoupling Theorem

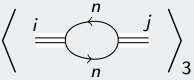
Theorem 14 (Recoupling Theorem)



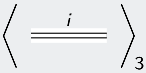
$$= \sum_{j=0}^{n} \begin{Bmatrix} n & n & (j,j) \\ n & n & (i,i) \end{Bmatrix}$$



Lemma 15



$$= \delta_{ij} \frac{\theta(n, n, (i,i))}{\Delta(i,i)}$$



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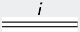
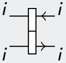
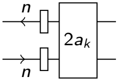
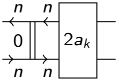
where 

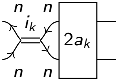
denotes 

and δ_{ij} is the Kronecker delta function.

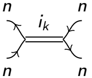
Proof of Theorem 2

$\Big\langle$ 

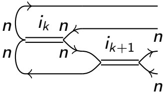
$\Big\rangle_3 =$ $\Big\langle$ 

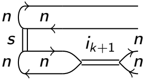
$\Big\rangle_3 = \displaystyle\sum_{j=0}^{n} \begin{Bmatrix} n & n & (i_k, i_k) \\ n & n & (0,0) \end{Bmatrix}$ $\Big\langle$ 

$\Big\rangle_3$

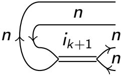
$= q^{-\frac{2a_k}{3}(n^2+3n)+a_k(i_k^2+2i_k)} \dfrac{\Delta(i_k, i_k)}{\theta(n, n, (i_k, i_k))}$ $\Big\langle$ 

$\Big\rangle_3.$

$\Big\langle$ 

$\Big\rangle_3 = \displaystyle\sum_{s=0}^{n} \begin{Bmatrix} n & n & (s, s) \\ n & n & (i_k, i_k) \end{Bmatrix}$ $\Big\langle$ 

$\Big\rangle_3$

$= \begin{Bmatrix} n & n & (i_{k+1}, i_{k+1}) \\ n & n & (i_k, i_k) \end{Bmatrix}$ $\Big\langle$ 

$\Big\rangle_3.$

The $\mathfrak{sl}_3$ tail of the $(2,2m)$-torus link

Definition 16

Suppose $f(q), f_n(q) \in \mathbb{Z}[[q]]$ for $n \geq 1$. The limit of $\{f_n(q)\}_n$ is $f(q)$, denoted $\lim_{n\to\infty} f_n(q) = f(q)$, means that $f_n(q) = f(q)$ in $\mathbb{Z}[[q]]/q^{n+1}\mathbb{Z}[[q]]$ for all n.

From Theorem 1,

$$J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m)) = q^{-\frac{2m}{3}(n^2+3n)+n} \sum_{0\leq k_m \leq \cdots \leq k_2 \leq k_1 \leq n} q^{-2k_m} q^{\sum_{j=1}^{m}(k_j^2+2k_j)} \times \frac{(q)_n^2}{(q)_{k_m}^2 (q)_{n-k_1}(q)_{k_1-k_2}\ldots(q)_{k_{m-1}-k_m}} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{n-k_m+1})(1-q^{n-k_m+2})}.$$

From Theorem 2,

$$J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m)) = q^{-\frac{2m}{3}(n^2+3n)+n} \sum_{i=0}^{n} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{(1-q)(1-q^{n+1})(1-q^{n+2})}.$$

$q^{\frac{2m}{3}(n^2+3n)-n} J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m))$ is in $\mathbb{Z}[[q]]$ and we can easily see the existence of the limit of $q^{\frac{2m}{3}(n^2+3n)-n} J^{\mathfrak{sl}_3}_{(n,0)}(T(2,2m))$. Consequently, we can obtaine the following q-series identity.

Theorem 17 (Y.)

$$\sum_{i=0}^{\infty} q^{-2i} q^{m(i^2+2i)} \frac{(1-q^{i+1})^3(1+q^{i+1})}{1-q}$$
$$= (q)_\infty \sum_{0 \le k_m \le \cdots \le k_2 \le k_1} \frac{q^{-2k_m} q^{\sum_{j=1}^{m}(k_i^2+2k_i)}}{(q)_{k_m}^2 (q)_{k_1-k_2} \ldots (q)_{k_{m-1}-k_m}}.$$

The above identity is a knot-theoretical generalization of the Andrews-Gordon identities for Ramanujan's false theta function.

$m = 1$

$$1 + O\left(q^{151}\right)$$

$m = 2$

$$
\begin{aligned}
&1 - q - q^2 + q^3 + q^4 + q^5 - q^6 - q^7 - q^8 - q^9 + q^{10} + q^{11} + q^{12} + q^{13} + q^{14} - q^{15} \\
&- q^{16} - q^{17} - q^{18} - q^{19} - q^{20} + q^{21} + q^{22} + q^{23} + q^{24} + q^{25} + q^{26} + q^{27} - q^{28} - q^{29} \\
&- q^{30} - q^{31} - q^{32} - q^{33} - q^{34} - q^{35} + q^{36} + q^{37} + q^{38} + q^{39} + q^{40} + q^{41} + q^{42} + q^{43} \\
&+ q^{44} - q^{45} - q^{46} - q^{47} - q^{48} - q^{49} - q^{50} - q^{51} - q^{52} - q^{53} - q^{54} + q^{55} + q^{56} + q^{57} \\
&+ q^{58} + q^{59} + q^{60} + q^{61} + q^{62} + q^{63} + q^{64} + q^{65} - q^{66} - q^{67} - q^{68} - q^{69} - q^{c}70 - q^{71} \\
&- q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} + q^{78} + q^{79} + q^{80} + q^{81} + q^{82} + q^{83} + q^{84} + q^{85} \\
&+ q^{86} + q^{87} + q^{88} + q^{89} + q^{90} - q^{91} - q^{92} - q^{93} - q^{94} - q^{95} - q^{96} - q^{97} - q^{98} - q^{99} \\
&- q^{100} - q^{101} - q^{102} - q^{103} - q^{104} + q^{105} + q^{106} + q^{107} + q^{108} + q^{109} + q^{110} + q^{111} \\
&+ q^{112} + q^{113} + q^{114} + q^{115} + q^{116} + q^{117} + q^{118} + q^{119} - q^{120} - q^{121} - q^{122} - q^{123} \\
&- q^{124} - q^{125} - q^{126} - q^{127} - q^{128} - q^{129} - q^{130} - q^{131} - q^{132} - q^{133} - q^{134} - q^{135} \\
&+ q^{136} + q^{137} + q^{138} + q^{139} + q^{140} + q^{141} + q^{142} + q^{143} + q^{144} + q^{145} + q^{146} + q^{147} \\
&+ q^{148} + q^{149} + q^{150} + O(q^{151})
\end{aligned}
$$

$m = 3$

$$
\begin{aligned}
&1 - q - q^2 + q^3 + q^7 + q^8 - q^9 - q^{10} - q^{11} - q^{12} + q^{13} + q^{14} + q^{20} + q^{21} + q^{22} - q^{23} \\
&- q^{24} - q^{25} - q^{26} - q^{27} - q^{28} + q^{29} + q^{30} + q^{31} + q^{39} + q^{40} + q^{41} + q^{42} - q^{43} - q^{44} \\
&- q^{45} - q^{46} - q^{47} - q^{48} - q^{49} - q^{50} + q^{51} + q^{52} + q^{53} + q^{54} + q^{64} + q^{65} + q^{66} + q^{67} \\
&+ q^{68} - q^{69} - q^{70} - q^{71} - q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} - q^{78} + q^{79} + q^{80} + q^{81} \\
&+ q^{82} + q^{83} + q^{95} + q^{96} + q^{97} + q^{98} + q^{99} + q^{100} - q^{101} - q^{102} - q^{103} - q^{104} - q^{105} \\
&- q^{106} - q^{107} - q^{108} - q^{109} - q^{110} - q^{111} - q^{112} + q^{113} + q^{114} + q^{115} + q^{116} + q^{117} \\
&+ q^{118} + q^{132} + q^{133} + q^{134} + q^{135} + q^{136} + q^{137} + q^{138} - q^{139} - q^{140} - q^{141} - q^{142} \\
&- q^{143} - q^{144} - q^{145} - q^{146} - q^{147} - q^{148} - q^{149} - q^{150} + O\left(q^{151}\right)
\end{aligned}
$$

$m = 4$

$$1 - q - q^2 + q^3 + q^{10} + q^{11} - q^{12} - q^{13} - q^{14} - q^{15} + q^{16} + q^{17} + q^{28} + q^{29} + q^{30} - q^{31} - q^{32} - q^{33} - q^{34} - q^{35} - q^{36} + q^{37} + q^{38} + q^{39} + q^{54} + q^{55} + q^{56} + q^{57} - q^{58} - q^{59} - q^{60} - q^{61} - q^{62} - q^{63} - q^{64} - q^{65} + q^{66} + q^{67} + q^{68} + q^{69} + q^{88} + q^{89} + q^{90} + q^{91} + q^{92} - q^{93} - q^{94} - q^{95} - q^{96} - q^{97} - q^{98} - q^{99} - q^{100} - q^{101} - q^{102} + q^{103} + q^{104} + q^{105} + q^{106} + q^{107} + q^{130} + q^{131} + q^{132} + q^{133} + q^{134} + q^{135} - q^{136} - q^{137} - q^{138} - q^{139} - q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} + q^{148} + q^{149} + q^{150} + O\left(q^{151}\right)$$

$m = 5$

$$1 - q - q^2 + q^3 + q^{13} + q^{14} - q^{15} - q^{16} - q^{17} - q^{18} + q^{19} + q^{20} + q^{36} + q^{37} + q^{38} - q^{39} - q^{40} - q^{41} - q^{42} - q^{43} - q^{44} + q^{45} + q^{46} + q^{47} + q^{69} + q^{70} + q^{71} + q^{72} - q^{73} - q^{74} - q^{75} - q^{76} - q^{77} - q^{78} - q^{79} - q^{80} + q^{81} + q^{82} + q^{83} + q^{84} + q^{112} + q^{113} + q^{114} + q^{115} + q^{116} - q^{117} - q^{118} - q^{119} - q^{120} - q^{121} - q^{122} - q^{123} - q^{124} - q^{125} - q^{126} + q^{127} + q^{128} + q^{129} + q^{130} + q^{131} + O\left(q^{151}\right)$$

$m = 6$

$$
\begin{aligned}
&1 - q - q^2 + q^3 + q^{16} + q^{17} - q^{18} - q^{19} - q^{20} - q^{21} + q^{22} + q^{23} + q^{44} + q^{45} + q^{46} \\
&- q^{47} - q^{48} - q^{49} - q^{50} - q^{51} - q^{52} + q^{53} + q^{54} + q^{55} + q^{84} + q^{85} + q^{86} + q^{87} - q^{88} \\
&- q^{89} - q^{90} - q^{91} - q^{92} - q^{93} - q^{94} - q^{95} + q^{96} + q^{97} + q^{98} + q^{99} + q^{136} + q^{137} \\
&+ q^{138} + q^{139} + q^{140} - q^{141} - q^{142} - q^{143} - q^{144} - q^{145} - q^{146} - q^{147} - q^{148} - q^{149} \\
&- q^{150} + O\left(q^{151}\right)
\end{aligned}
$$

Visions of our formulas

- Computation of trihedron coefficients of $\mathcal{U}_q(\mathfrak{sl}_3)$ in general.
- Computation of colored $\mathfrak{sl}_3$ Jones polynomials with weight (m,n).
- Colored version of polynomials of oriented surface-link diagrams defined by Joung, Kamada, Kawauchi and Lee.
- Quantum $SU(3)$ representation of the mapping class group of a surface.

FIN

Thank you for your attention!