Topology of the Milnor fibrations of polar weighted homogeneous polynomials

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§1. Mixed polynomials

We study a polynomial of complex variables and complex conjugates:

$$P(\mathbf{z},\bar{\mathbf{z}}) := \sum_{i=1}^{m} c_i \mathbf{z}^{\nu_i}\bar{\mathbf{z}}^{\mu_i},$$

where $c_i \in \mathbb{C}^*$ and $\mathbf{z}^{\nu_i} = z_1^{\nu_{i,1}} \cdots z_n^{\nu_{i,n}}$ for $\nu_i = (\nu_{i,1}, \dots, \nu_{i,n})$ (respectively $\bar{\mathbf{z}}^{\mu_i} = \bar{z}_1^{\mu_{i,1}} \cdots \bar{z}_n^{\mu_{i,n}}$ for $\mu_i = (\mu_{i,1}, \dots, \mu_{i,n})$).
We call P a mixed polynomial.

A mixed polynomial map P is a map from $\mathbb{R}^{2n}$ to $\mathbb{R}^2$:

$$\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right) \mapsto \left(\Re P\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right), \Im P\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right)\right).$$

A point $\mathbf{w} \in \mathbb{C}^n$ is called a mixed singular point of P if the gradient vectors of $\Re P$ and $\Im P$ are linearly dependent at $\mathbf{w}$.

Suppose that $P(0, \ldots, 0) = 0$ and $\mathbf{o} = (0, \ldots, 0)$ is a mixed singular point of P. If there exist positive real numbers ε and δ with $\delta \ll \varepsilon \ll 1$ such that the map

$$P : D^{2n}_{\varepsilon} \cap P^{-1}(\partial D^2_{\delta}) \to \partial D^2_{\delta}$$

is a locally trivial fibration over ∂D^2_{δ}, We say that this map is called the Milnor fibration of P at the origin, where
$D^{2n}_{\varepsilon} = \{\mathbf{z} \in \mathbb{C}^n \mid \|\mathbf{z}\| \leq \varepsilon\}$ and $D^2_{\delta} = \{\eta \in \mathbb{C} \mid |\eta| \leq \delta\}$.

Let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be integers such that $\gcd(p_1, \dots, p_n) = \gcd(q_1, \dots, q_n) = 1$. We define the S^1-action and the $\mathbb{R}^*$-action on $\mathbb{C}^n$ as follows:

$$s \circ \mathrm{z} = (s^{p_1} z_1, \dots, s^{p_n} z_n), \quad s \in S^1,$$
$$r \circ \mathrm{z} = (r^{q_1} z_1, \dots, r^{q_n} z_n), \quad r \in \mathbb{R}^*.$$

If there exists a positive integer d_p such that $P(\mathrm{z}, \bar{\mathrm{z}})$ satisfies

$$P(s^{p_1} z_1, \dots, s^{p_n} z_n, \bar{s}^{p_1} \bar{z}_1, \dots, \bar{s}^{p_1} \bar{z}_n) = s^{d_p} P(\mathrm{z}, \bar{\mathrm{z}}), \quad s \in S^1,$$

we say that $P(\mathrm{z}, \bar{\mathrm{z}})$ is a polar weighted homogeneous polynomial.

Similarly $P(\mathrm{z}, \bar{\mathrm{z}})$ is called a radial weighted homogeneous polynomial if there exists a positive integer d_r such that

$$P(r^{q_1} z_1, \ldots, r^{q_n} z_n, r^{q_1} \bar{z}_1, \ldots, r^{q_n} \bar{z}_n) = r^{d_r} P(\mathrm{z}, \bar{\mathrm{z}}), \quad r \in \mathbb{R}^*.$$

Let P be a polar and radial weighted homogeneous polynomial. Then P admits the global Milnor fibration

$$P : \mathbb{C}^n \setminus P^{-1}(0) \to \mathbb{C}^*$$

[Ruas-Seade-Verjovsky, Cisneros-Molina, Oka].

The monodromy map $h : F \to F$ is given by

$$h(\mathrm{z}) = \Big(\exp\Big(\frac{2p_1\pi}{d_p}\Big) z_1, \ldots, \exp\Big(\frac{2p_n\pi}{d_p}\Big) z_n \Big),$$

where $F = P^{-1}(1)$.

Theorem [Oka 2010].

Let P be a polar and radial weighted homogeneous polynomial. Then the two fibrations

$$P : D_\varepsilon^{2n} \cap P^{-1}(\partial D_\delta^2) \to \partial D_\delta^2, \quad \frac{P}{|P|} : S_\varepsilon^{2n-1} \setminus \mathrm{Int}N(K_P) \to S^1$$

are isomorphic, where $K_P = S_\varepsilon^{2n-1} \cap P^{-1}(0)$.

If the origin is an isolated singularity of P, we call K_P the link of P at the origin.

Let P be a 2-variable polar weighted homogeneous polynomial which has an isolated singularity at the origin. Then the link of P at the origin is a Seifert link, denoted by $L(P, \mathrm{o})$.

A connected component of $L(P, \mathrm{o})$ is called a positive component if the orientation of the link component coincides with that of the S^1-action, and otherwise it is called a negative component.

Let $|L^+(P, \mathrm{o})|$ and $|L^-(P, \mathrm{o})|$ denote the number of positive components of $L(P, \mathrm{o})$ and the number of negative components of $L(P, \mathrm{o})$, respectively.

§2. Deformations

A deformation of P is a polynomial map

$$F : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}, (\mathbf{z}, t) \mapsto F_t(\mathbf{z}),$$

with $F_0(\mathbf{z}) = P(\mathbf{z}, \bar{\mathbf{z}})$.

For complex isolated singularities, there exist a deformation F_t of a complex polynomial $P(\mathbf{z})$ such that any singularity of $F_t(\mathbf{z})$ is a Morse singularity for any $0 < t \ll 1$.

A sufficiently small compact neighborhood of each Morse singularity can be regarded as a $2n$-dimensional n-handle and we have a decomposition

$$D_\varepsilon^{2n} \cap F_t^{-1}(D_\delta^2) \cong \left(D_\varepsilon^{2n} \cap F_t^{-1}(D_{\delta_t}^2)\right) \cup_\varphi \left(\sqcup_{i=1}^{\ell} (n\text{-handle})_i \right),$$

where ℓ is the Milnor number of the singularity of P at o, $\varphi = (\varphi_1, \ldots \varphi_\ell)$ is the ℓ-tuple of the attaching map φ_i of $(n\text{-handle})_i$ and $D_{\delta_t}^2$ is a 2-disk centered at 0 with radius δ_t such that $\delta_t < \delta$ and $F_t \mid_{F_t^{-1}(D_{\delta_t}^2)}$ has no singularities.

Let P be a 2-variable polar and radial weighted homogeneous polynomial which has an isolated singularity at the origin of $\mathbb{C}^2$.

Example.

The mixed polynomial $P = f\bar{g}$ is polar and radial weighted homogeneous, where $f(\mathrm{z})$ and $g(\mathrm{z})$ can be written as

$$f(\mathrm{z}) = \prod_{j=1}^{m}(z_1^p + \alpha_j z_2^q), \quad g(\mathrm{z}) = \prod_{j=1}^{n}(z_1^p + \beta_j z_2^q), \quad \gcd(p,q) = 1,$$

$\alpha_j \neq \alpha_{j'}, \beta_j \neq \beta_{j'}$ $(j \neq j')$, $\alpha_k \neq \beta_{k'}$ for $1 \leq k \leq m$ and $1 \leq k' \leq n$.

Let F_t be a deformation of P. Set $S_k(F_t) = \{\mathrm{z} \in U \mid \text{rank}\, dF_t(\mathrm{z}) = 2 - k\}$ for $k = 1, 2$.

We assume that F_t satisfies the following properties:

1. F_t is polar weighted homogeneous for any $0 \leq t \ll 1$;
2. For each point $\mathrm{w} \in S_1(F_t)$, there exist local coordinates (x_1, x_2, x_3, x_4) centered at w such that F_t is given by

$$(F_t/|F_t|, |F_t|) = (x_1, -x_2^2 + x_3^2 + x_4^2 + c_{t,\mathrm{w}}),$$

   where $c_{t,\mathrm{w}} = |F_t(\mathrm{w})|$ for $\mathrm{w} \in S_1(F_t)$ and $0 < t \ll 1$;
3. $S_2(F_t) = \{\mathrm{o}\}$ or $\emptyset$.

Theorem [I 2016].

Let f and g be 2-variable weighted homogeneous complex polynomials which have no common branches. Then there exists a deformation of F_t of $f\bar{g}$ such that F_t satisfies the above conditions.

Round handles [Asimov]

Let X and Y be n-dimensional smooth manifolds. We say that X is obtained from Y by attaching a round k-handle if

1. there are disk bundles over S^1, E_s^k and E_u^{n-k-1},
2. there exists an embedding $\varphi : \partial E_s^k \times_{S^1} E_u^{n-k-1} \longrightarrow \partial Y$ such that $X \cong Y \cup_\phi E_s^k \oplus E_u^{n-k-1}$,

where $E_s^k \oplus E_u^{n-k-1}$ is the Whitney sum of E_s^k and E_u^{n-k-1} over S^1. The bundle $E_s^k \oplus E_u^{n-k-1}$ over S^1 is called an n-dimensional round k-handle and φ is called the attaching map of $E_s^k \oplus E_u^{n-k-1}$.

Note that a sufficiently small compact neighborhood of each connected component of $S_1(F_t)$ is regarded as a 4-dimensional round 1-handle.

By the condition (2), $|F_t|$ defines a Morse function on $(D^4_\varepsilon \cap F_t^{-1}(D^2_\delta))/S^1$ for $t > 0$ and outside the image of the origin, and the indices of the Morse singularities are always 1.

Then the handle decomposition of the orbit space according to this Morse function induces the following decomposition:

$$D^4_\varepsilon \cap F_t^{-1}(D^2_\delta) \cong \big(D^4_\varepsilon \cap F_t^{-1}(D^2_{\delta_t})\big) \cup_\varphi \big(\sqcup_{i=1}^{\ell} \text{(round 1-handle)}_i \big)$$

where $\varphi = (\varphi_1, \dots \varphi_\ell)$ is the attaching map of ℓ copies of a round 1-handle. Here we may assume that the images of $\varphi_1, \dots, \varphi_\ell$ in $\partial(D^4_\varepsilon \cap F_t^{-1}(D^2_{\delta_t}))$ are disjoint.

§3. Main result

Theorem [I 2016].

Let P be a 2-variable polar and radial weighted homogeneous polynomial which has an isolated singularity at the origin and let F_t be a deformation of P satisfying the conditions $(1), (2)$ and (3) for $0 < t \ll 1$. Then

1. $D^4_\varepsilon \cap F_t^{-1}(D^2_{\delta_t})$ is diffeomorphic to the disjoint union of a 4-ball and ℓ copies of $S^1 \times B^3$, where B^3 is a 3-ball, and each φ_i of the attaching map $\varphi = (\varphi_1, \dots, \varphi_\ell)$ maps the two attaching regions of the i-th round 1-handle to both of the boundary of the 4-ball and that of the i-th $S^1 \times B^3$; and
2. the number ℓ of round 1-handles is given as

$$\ell = |L^+(P, \mathrm{o})| - |L^+(F_t, \mathrm{o})| = |L^-(P, \mathrm{o})| - |L^-(F_t, \mathrm{o})|.$$

Sketch of the proof

Put $M_0 = D^4_\varepsilon \cap F_t^{-1}(D^2_{\delta_t}) = \sqcup_{j=0}^{\ell} M_0^j$. Assume that $\mathrm{o} \in M_0^0$. We consider the restricted Milnor fibration
$F_t : D^4_\varepsilon \cap F_t^{-1}(\partial D^2_{\delta_t}) \to \partial D^2_{\delta_t}$ and connected components of M_0.

Lemma 1.

Let S_0 be the fiber surface of $F_t : D^4_\varepsilon \cap F_t^{-1}(\partial D^2_{\delta_t}) \to \partial D^2_{\delta_t}$. Then S_0 is diffeomorphic to the disjoint union of the fiber surface of $F_t \mid_{D^4_{\varepsilon_t} \cap F_t^{-1}(\partial D^2_{\delta_t})}$ and ℓ copies of an annulus, where ℓ is the number of connected components of $S_1(F_t)$, where $0 < \varepsilon_t \ll \varepsilon$.

Lemma 2.

The connected component M_0^0 of M_0 is diffeomorphic to a 4-ball and M_0^j is diffeomorphic to $S^1 \times B^3$, where B^3 is a 3-ball, for $j = 1, \ldots, \ell$.

Lemma 3.

The orbit space of $D^4_\varepsilon \cap F_t^{-1}(\partial D^2_\delta)$ of the S^1-action is homeomorphic to a holed 2-sphere for $0 \leq t \ll 1$.

By the condition (2), the absolute value $|F_t|$ of F_t defines a Morse function on $(D^4_\varepsilon \cap F_t^{-1}(D^2_\delta))/S^1$ for $t > 0$ and outside the image of the origin, and the indices of the Morse singularities are always 1. By using Lemma 3, we have the following Lemma.

Lemma 4.

Let N_i be a connected component of a sufficiently small neighborhood of $S_1(F_t)$ such that N_i/S^1 is a 3-dimensional 1-handle for $i = 1, \ldots, \ell$. Then N_i connects two connected components of M_0.

Set $\tilde{M}_0 = D^4_\varepsilon \cap F_t^{-1}(\partial D^2_{\delta_t})$, $\tilde{M}_i = \tilde{M}_{i-1} \cup_{\varphi_i} \partial N_i$ and S_i is the fiber surface of $F_t \mid_{\tilde{M}_i}$ for $i = 1, \ldots, \ell$.
Since a fiber of $F_t : N_i \longrightarrow D^2_\delta$ is a d_p-fold cover over a fiber of $|F_t|$, we have

$$F_t^{-1}(u) \cap N_i \cong \begin{cases} (\sqcup_{j=1}^{d_p} D^2_{1,j}) \sqcup (\sqcup_{j=1}^{d_p} D^2_{2,j}) & 0 < c_{t,\mathrm{w}} - |u| \ll 1 \\ \sqcup_{j=1}^{d_p} A_j & 0 < |u| - c_{t,\mathrm{w}} \ll 1 \end{cases},$$

where $D^2_{k,j}$ is a 2-disk and A_j is an annulus for $k = 1, 2$ and $j = 1, \ldots, d_p$. Thus $\chi(S_i) - \chi(S_{i-1})$ is equal to $-2d_p$.

Lemma 5.

Let ℓ be the number of connected components of $S_1(F_t)$. Then ℓ is equal to $|L^+(P, \mathrm{o})| - |L^+(F_t, \mathrm{o})|$ and also to $|L^-(P, \mathrm{o})| - |L^-(F_t, \mathrm{o})|$.

§4. Monodromy

Let $h_i : S_i \longrightarrow S_i$ be the monodromy of $F_t \mid_{\tilde{M}_i}$ for $i = 1, \ldots, \ell$. Since h_i is given by the S^1-action on $\mathbb{C}^2$, $h_i : S_i \longrightarrow S_i$ satisfies the following conditions:

1. $h_i(S_{i-1} \setminus (D'_1 \sqcup D'_2)) = S_{i-1} \setminus (D'_1 \sqcup D'_2)$ and $h_i \mid_{S_{i-1}\setminus(D'_1 \sqcup D'_2)} = h_{i-1} \mid_{S_{i-1}\setminus(D'_1 \sqcup D'_2)}$,
2. $h_i \mid_{D'_k}$ and $h_i \mid_{A'}$ are periodic maps which satisfy $D^2_{k,j} \longrightarrow D^2_{k,j+1}$ and $A_j \longrightarrow A_{j+1}$,

where $A' = \sqcup_{j=1}^{d_p} A_j, D'_k = \sqcup_{j=1}^{d_p} D^2_{k.j}$ for $i = 1, \ldots, \ell, j = 1, \ldots, d_p$ and $k = 1, 2$. Here $D^2_{k,d_p+1} = D^2_{k,1}$ and $A_{d_p+1} = A_1$.

Let $h : S \to S$ be a homeomorphism of a surface S. We define

$$\Delta_*(h) = \frac{\Delta_1(h)}{\Delta_0(h)},$$

where $\Delta_i(h)$ is the characteristic polynomial of the homological map from $H_i(S, \mathbb{Z})$ to itself induced by h for $i = 0, 1$.

Lemma 6.

Let S_i be the fiber surface of $F_t \mid_{\tilde{M}_i}$ and $h_i : S_i \to S_i$ be the monodromy of $F_t \mid_{\tilde{M}_i}$ for $i = 1, \ldots, \ell$. Then the characteristic polynomial of h_i satisfies that $\Delta_*(h_i) = \Delta_*(h_{i-1})(t^{d_p} - 1)^2$.

Theorem [I 2016].

Let h be the monodromy of $P : D_\varepsilon^4 \cap P^{-1}(\partial D_\delta^2) \to \partial D_\delta^2$. Then $\Delta_*(h)$ is equal to $\Delta_*(h_0)(t^{d_p} - 1)^{2\ell}$.

Thank you for your attention!