Branched twist spins and knot determinants from the point of view of representations

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Outline

§1. Twist spun knots

Notations.

$h : S^n \hookrightarrow S^{n+2}$: a smooth embedding

$K := h(S^n)$: an n-knot

$K \sim K' \Leftrightarrow \exists F_t : S^{n+2} \to S^{n+2}$: a smooth isotopy
s.t. $F_0 = \mathrm{id}, F_1(K) = K'$.

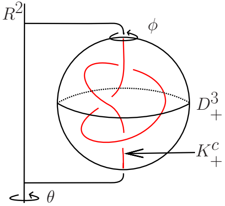
$E(K)$: the knot exterior of K

Zeeman's construction ('63)

- K : 1-knot in S^3
- $R^3_+ := \{(x_1, x_2, x_3) \in R^3 \mid x_3 \geq 0\} \subset R^3 \subset S^3$

Set K in R^3_+ s.t. $K \cap \partial R^3_+$ is an interval.

- $D^3_+ \subset R^3_+ \setminus \partial R^3_+$: a 3-ball
- $K^c_+ := D^3_+ \cap K_+$



Definition (Twist spun knot)

The $m-$twist spun knot of K is defined as
$K^m := K^c_+ \times \partial D^2 \cup_{f_m} \partial K^c_+ \times D^2$.

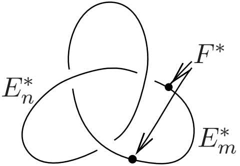
Theorem (Zeeman '63)

$\pi_1(E(K^m)) \cong \pi_1(E(K))/\langle \theta^m : \mathrm{central} \rangle$,
where θ is an element induced by a meridian of K.

§2. Branched twist spins and known results

Definition of branched twist spins.

- $S^1 \curvearrowright S^4$: an effective locally smooth action
- $p : S^4 \to S^3$: the projection
- E_m, E_n : the sets of exceptional orbits
  $p(E_m) := E_m^*$, $p(E_n) := E_n^*$
- F : the fixed point set, $p(F) := F^*$



Definition (Branched twist spin)

Let K be the 1-knot $E_m^* \cup E_n^* \cup F^*$. The (m, n)-branched twist spin of K is defined as $K^{m,n} := E_n \cup F$.

Remark

The 2-knot $K^{m,1}$ is the m-twist spun knot of K.

Known results

ε : the sign of m

Theorem (Plotnick '84, F. '16)

Let $K^{m,n}$ be the branched twist spin of K and let β be an integer such that $n\beta \equiv \varepsilon \pmod{m}$. Then,

$$\pi_1(E(K^{m,n})) \cong \pi_1(E(K)) * \langle h \rangle / \langle \theta^{|m|} h^{\beta} = 1, h : \mathrm{central} \rangle,$$

where θ is an element induced by a meridian of K.

Theorem (Hillman-Plotnick '90)

Let K be a torus or hyperbolic knot.
Then $K^{m,n}$ is non-trivial if $m > n$ and $m \geq 3$.

Main results

Theorem1 (F. '16)

Let $K_1^{m_1,n_1}, K_2^{m_2,n_2}$ be branched twist spins constructed from 1-knots K_1, K_2, respectively.
(1) If m_1, m_2 are even,

$$|\Delta_{K_1}(-1)| \neq |\Delta_{K_2}(-1)| \Rightarrow K_1^{m_1,n_1} \not\sim K_2^{m_2,n_2}.$$

(2) If m_1 is even and m_2 is odd,

$$|\Delta_{K_1}(-1)| \neq 1 \Rightarrow K_1^{m_1,n_1} \not\sim K_2^{m_2,n_2}.$$

Proof of Theorem1 (a property of $E_k(K^{m,n})$)

Recall

Consider an S^1-action on S^4. The image of singular orbits constitutes a 1-knot $E_m^* \cup E_n^* \cup F^*$ in S^3. Then $K^{m,n}$ is defined as $E_n \cup F$.

Plotnick's theorem says

$$\pi_1(E(K^{m,n})) \cong \pi_1(E(K)) * \langle h \rangle / \langle \theta^{|m|} h^\beta = 1, h : \mathrm{central} \rangle.$$

Wirtinger presentation gives

$$\pi_1(E(K^{m,n})) \cong \langle x_1, \ldots, x_l, h \mid r_1, \ldots, r_l, x_i h x_i^{-1} h^{-1}, x_1^{|m|} h^\beta \rangle.$$

Proof of Main theorem(a property of $E_k(K^{m,n})$)

The Alexander matrix of $K^{m,n}$ is given as

$$\begin{pmatrix} & & & & 0 \\ & & & & \vdots \\ & B_K & & & \vdots \\ & & & & \vdots \\ & & & & 0 \\ 1-t^m & & & & t^{-\beta}-1 \\ & \ddots & & O & \vdots \\ O & & \ddots & & \vdots \\ & & & 1-t^m & t^{-\beta}-1 \\ \frac{1-t^{-m\beta}}{1-t^{-\beta}} & 0 & \dots & 0 & \frac{t^{-m\beta}(1-t^{m\beta})}{1-t^m} \end{pmatrix}$$

Proof of Main theorem(a property of $E_k(K^{m,n})$)

$$
\begin{pmatrix}
0 & & & & 0 \\
\vdots & & & & \vdots \\
\vdots & & B'_K & & \vdots \\
\vdots & & & & \vdots \\
0 & & & & 0 \\
1 - t^m & & & & t^{-\beta} - 1 \\
 & \ddots & & O & 0 \\
 & & \ddots & & \vdots \\
O & & & & \vdots \\
 & & & 1 - t^m & 0 \\
\frac{1-t^{-m\beta}}{1-t^{-\beta}} & 0 & \cdots & 0 & \frac{t^{-m\beta}(1-t^{m\beta})}{1-t^m}
\end{pmatrix}
$$

Then the 0-th elementary ideal $E_0(K^{m,n}) = 0$.

Proof of Theorem1 (a property of $E_k(K^{m,n})$)

Lemma (F. '16)

The 1st elementary ideal $E_1(K^{m,n})$ is the ideal generated by the following elements:

$$\begin{cases} \Delta_K(t^\beta)\left\{(1-t^{|m|}),(1-t^\beta),\frac{1-t^{|m|\beta}}{1-t^\beta},\frac{1-t^{|m|\beta}}{1-t^{|m|}}\right\}, \\ G_i(t^\beta)(1-t^{|m|})\left\{(1-t^{|m|}),(1-t^\beta),\frac{1-t^{|m|\beta}}{1-t^\beta},\frac{1-t^{|m|\beta}}{1-t^{|m|}}\right\}, \\ (1-t^{|m|})^{l-1}\left\{(1-t^{|m|}),(1-t^\beta),\frac{1-t^{|m|\beta}}{1-t^\beta},\frac{1-t^{|m|\beta}}{1-t^{|m|}}\right\}. \end{cases}$$

Especially, $E_1(K^{m,n}) \neq 0$.

Here $P\{Q_1,Q_2,Q_3,Q_4\}$ means PQ_1,PQ_2,PQ_3,PQ_4.

Proof of Theorem1 (compering generators)

Here, we consider the contraposition of Main theorem. Suppose that $K_1^{m_1,n_1} \sim K_2^{m_2,n_2}$.
The generators of $E_1(K_i^{m_i,n_i})$ are written as

$$E_1(K_1^{m_1,n_1}) \begin{cases} \Delta_K(t^{\beta_1})\left\{(1-t^{|m_1|}),(1-t^{\beta_1}),\dfrac{1-t^{|m_1|\beta_1}}{1-t^{\beta_1}},\dfrac{1-t^{|m_1|\beta_1}}{1-t^{|m_1|}}\right\}, \\ G_1(t^{\beta_1})(1-t^{|m_1|})\left\{(1-t^{|m_1|}),(1-t^{\beta_1}),\dfrac{1-t^{|m_1|\beta_1}}{1-t^{\beta_1}},\dfrac{1-t^{|m_1|\beta_1}}{1-t^{|m_1|}}\right\}, \\ (1-t^{|m_1|})^{l-1}\left\{(1-t^{|m_1|}),(1-t^{\beta_1}),\dfrac{1-t^{|m_1|\beta_1}}{1-t^{\beta_1}},\dfrac{1-t^{|m_1|\beta_1}}{1-t^{|m_1|}}\right\}. \end{cases}$$

$$E_1(K_2^{m_2,n_2}) \begin{cases} \Delta_K(t^{\beta_2})\left\{(1-t^{|m_2|}),(1-t^{\beta_2}),\dfrac{1-t^{|m_2|\beta_2}}{1-t^{\beta_2}},\dfrac{1-t^{|m_2|\beta_2}}{1-t^{|m_2|}}\right\}, \\ G_2(t^{\beta_2})(1-t^{|m_2|})\left\{(1-t^{|m_2|}),(1-t^{\beta_2}),\dfrac{1-t^{|m_2|\beta_2}}{1-t^{\beta_2}},\dfrac{1-t^{|m_2|\beta_2}}{1-t^{|m_2|}}\right\}, \\ (1-t^{|m_2|})^{l-1}\left\{(1-t^{|m_2|}),(1-t^{\beta_2}),\dfrac{1-t^{|m_2|\beta_2}}{1-t^{\beta_2}},\dfrac{1-t^{|m_2|\beta_2}}{1-t^{|m_2|}}\right\}. \end{cases}$$

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$$E_1(K_2^{m_2,n_2}) \begin{cases} \Delta_K(t^{\beta_2})\left\{(1-t^{|m_2|}),(1-t^{\beta_2}),\frac{1-t^{|m_2|\beta_2}}{1-t^{\beta_2}},\frac{1-t^{|m_2|\beta_2}}{1-t^{|m_2|}}\right\}, \\ G_2(t^{\beta_2})(1-t^{|m_2|})\left\{(1-t^{|m_2|}),(1-t^{\beta_2}),\frac{1-t^{|m_2|\beta_2}}{1-t^{\beta_2}},\frac{1-t^{|m_2|\beta_2}}{1-t^{|m_2|}}\right\}, \\ (1-t^{|m_2|})^{l-1}\left\{(1-t^{|m_2|}),(1-t^{\beta_2}),\frac{1-t^{|m_2|\beta_2}}{1-t^{\beta_2}},\frac{1-t^{|m_2|\beta_2}}{1-t^{|m_2|}}\right\}. \end{cases}$$

Proof of Theorem1 (comparing generators)

For example, The following equality holds:

$$\begin{aligned}
&\Delta_{K_2}(t^{\beta_2})(1-t^{|m_2|})\\
&=\Delta_{K_1}(t^{\beta_1})\times\left\{P_1(t)(1-t^{|m_1|})+P_2(t)(1-t^{\beta_1})+P_3(t)\frac{1-t^{|m_1|\beta_1}}{1-t^{\beta_1}}+P_4(t)\frac{1-t^{|m_1|\beta_1}}{1-t^{|m_1|}}\right\}\\
&\quad+\sum_j G_j^1(t^{\beta_1})(1-t^{|m_1|})\times\Bigg\{P_5^j(t)(1-t^{|m_1|})+P_6^j(t)(1-t^{\beta_1})\\
&\quad+P_7^j(t)\frac{1-t^{|m_1|\beta_1}}{1-t^{\beta_1}}+P_8^j(t)\frac{1-t^{|m_1|\beta_1}}{1-t^{|m_1|}}\Bigg\}+(1-t^{|m_1|})^{l_1-1}\Bigg\{P_9(t)(1-t^{|m_1|})\\
&\quad+P_{10}(t)(1-t^{\beta_1})+P_{11}(t)\frac{1-t^{|m_1|\beta_1}}{1-t^{\beta_1}}+P_{12}(t)\frac{1-t^{|m_1|\beta_1}}{1-t^{|m_1|}}\Bigg\}.
\end{aligned}$$

Now, we use the condition m_1 is even. Substituting -1 for t in this equality, we have

$$\Delta_{K_2}((-1)^{\beta_2})(1-(-1)^{|m_2|})=\Delta_{K_1}((-1)^{\beta_1})(2P_2(-1)+\beta_1 P_4(-1)).$$

Proof of Theorem1 (comparing generators)

$$\Delta_{K_2}((-1)^{\beta_2})(1-(-1)^{|m_2|}) = \Delta_{K_1}((-1)^{\beta_1})(2P_2(-1)+\beta_1 P_4(-1)).$$

If m_2 is odd,

$$\frac{\Delta_{K_2}((-1)^{\beta_2})}{\Delta_{K_1}((-1)^{\beta_1})} = \frac{2P_2(-1)+\beta_1 P_4(-1)}{1-(-1)^{|m_2|}} = \frac{2P_2(-1)+\beta_1 P_4(-1)}{2} \in \frac{\mathbb{Z}}{2}.$$

The same aguments for other generators give the following table.

Generators	$1-t^{\lvert m_2\rvert}$	$1-t^{\beta_2}$	$\frac{1-t^{\lvert m_2\rvert\beta_2}}{1-t^{\beta_2}}$	$\frac{1-t^{\lvert m_2\rvert\beta_2}}{1-t^{\lvert m_2\rvert}}$
(1) $(m_2,\beta_1,\beta_2) =$ (e,o,o)		$\frac{\mathbb{Z}}{2}$		$\frac{\mathbb{Z}}{\beta_2} \ni \frac{\Delta_{K_2}}{\Delta_{K_1}}$
(2) $(m_2,\beta_1,\beta_2) =$ (o,o,e)	$\frac{\mathbb{Z}}{2}$		$\frac{\mathbb{Z}}{\lvert m_2\rvert}$	

Proof of Theorem 1 (comparing generators)

Generators	$1-t^{\lvert m_2\rvert}$	$1-t^{\beta_2}$	$\frac{1-t^{\lvert m_2\rvert\beta_2}}{1-t^{\beta_2}}$	$\frac{1-t^{\lvert m_2\rvert\beta_2}}{1-t^{\lvert m_2\rvert}}$
(1) $(m_2,\beta_1,\beta_2) =$ (e,o,o)		$\frac{\mathbb{Z}}{2}$		$\frac{\mathbb{Z}}{\beta_2} \ni \frac{\Delta_{K_2}}{\Delta_{K_1}}$

In the case (1),

$$\frac{\Delta_{K_2}((-1)^{\beta_2})}{\Delta_{K_1}((-1)^{\beta_1})} = \frac{\Delta_{K_2}(-1)}{\Delta_{K_1}(-1)} \in \mathbb{Z}.$$

Since m_1 and m_2 are even, switching the role of K_1 and K_2, we have

$$\frac{\Delta_{K_1}(-1)}{\Delta_{K_2}(-1)} \in \mathbb{Z}.$$

Therefore,

$$|\Delta_{K_1}(-1)| = |\Delta_{K_2}(-1)|.$$

Proof of Theorem 1 (comparing generators)

Generators	$1-t^{\lvert m_2\rvert}$	$1-t^{\beta_2}$	$\frac{1-t^{\lvert m_2\rvert\beta_2}}{1-t^{\beta_2}}$	$\frac{1-t^{\lvert m_2\rvert\beta_2}}{1-t^{\lvert m_2\rvert}}$
(2) $(m_2, \beta_1, \beta_2) =$ (o,o,e)	$\frac{\mathbb{Z}}{2}$		$\frac{\mathbb{Z}}{\lvert m_2\rvert} \ni \frac{\Delta_{K_2}}{\Delta_{K_1}}$	

In the case (2),

$$\frac{\Delta_{K_2}((-1)^{\beta_2})}{\Delta_{K_1}((-1)^{\beta_1})} = \frac{\Delta_{K_2}(1)}{\Delta_{K_1}(-1)} \in \mathbb{Z}.$$

For any 1-knot K, $\Delta_K(1) = 1$. Then,

$$\frac{1}{\Delta_{K_1}(-1)} \in \mathbb{Z}.$$

Therefore, we have

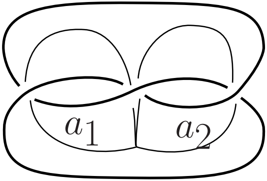
$$|\Delta_{K_1}(-1)| = 1.$$

□

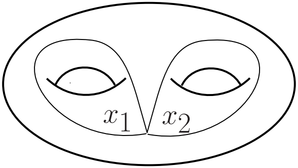
§4 Metabelian representations

Lin's presentation

Let K be a 1-knot and S be a Seifert surface for K of genus g.
$H_1 := S \times I,\ \pi_1(H_1) \cong \langle a_1, \ldots, a_{2g} \rangle$
$H_2 := S^3 \setminus \mathrm{int} H_1,\ \pi_1(H_2) \cong \langle x_1, \ldots, x_{2g} \rangle$



S



H_2

Lin's presentation

Theorem (Lin '01)

$$\pi_1(E(K)) \cong \langle x_1, \ldots, x_{2g}, \mu \mid \mu\alpha_i\mu^{-1} = \beta_i \ (i = 1, \ldots, 2g)\rangle,$$

where μ is the element induced a meridian of K and α_i, β_i are the word in $x_1, \ldots, x_{2g}$ determined by the attaching map of H_1 and H_2.

$\rho : G \to F$: a representation
ρ : metabelian $\Leftrightarrow$ $\rho([G, G])$ is abelian

Theorem (Lin '01)

The number of irreducible $SU(2, \mathbb{C})$-metaberian representations of K is

$$\frac{\mid \Delta_K(-1) - 1 \mid}{2}.$$

Representation of $K^{m,n}$

M_K : m-fold cyclic cover of K

Theorem (Plotnick '86)

Let $K^{m,n}$ be the branched twist spin of K. Then,

$$\pi_1(E(K^{m,n})) \cong \pi_1(M_{K^{m,n}}) * \langle \eta \rangle / \langle \eta z \eta^{-1} = \tau^n z \rangle,$$

where η represents a meridian of $K^{m,n}$ and $z \in \pi_1(M_{K^{m,n}})$.

Theorem2 (F. '16)

The number of irr. $SU(2,\mathbb{C})$-metabelian representation of $K^{m,n}$ is

$$\begin{cases} \dfrac{|\Delta_K(-1) - 1|}{2} & (m : \text{even}) \\ 0 & (m : \text{odd}). \end{cases}$$

Sketch of proof(1/3)

$K^{m,n}$ is a fibered knot whose fiber is puncM_K.
$\langle x_1, \dots, x_{2g}, \mu \mid \mu\alpha_i\mu^{-1} = \beta_i \ (i = 1, \dots, 2g)\rangle$: Lin's presentation
$\tilde{x}_i$: the lift of x_i to M_K
τ : covering transformation of M_K

Theorem (Nagasato-Yamaguchi '08)

The fundamental group of M_K is given as

$$\langle \tau^0\tilde{x}_1, \dots, \tau^0\tilde{x}_{2g}, \dots, \tau^{m-1}\tilde{x}_1, \dots \tau^{m-1}\tilde{x}_{2g} \mid \tilde{\alpha}_i^{(j)} = \tilde{\beta}_i^{(j-1)}\rangle,$$

where $\tilde{\alpha}_i^{(j)}$, $\tilde{\beta}_i^{(j)}$ denote the word obtained from replacing $\tilde{x}_1, \dots \tilde{x}_{2g}$ to $\tau^j\tilde{x}_1, \dots \tau^j\tilde{x}_{2g}$ for $i = 1, \dots, 2g$ and $j = 0, \dots, m-1$.

Sketch of proof(2/3)

By Plotnick's theorem, we have

$$\pi_1(E(K^{m,n})) \cong \pi_1(M_K) * \langle \eta \rangle / \langle \eta z \eta^{-1} = \tau^n z) \rangle.$$

Nagasato-Yamaguchi's and Lin's presentations give

$$\pi_1(E(K^{m,n})) \cong \langle \tau^j \tilde{x}_i, \eta \mid \tilde{\alpha}_i^{(j)} = \tilde{\beta}_i^{(j-1)}, \eta \tau^j \tilde{x}_i \eta^{-1} = \tau^{j+n} \tilde{x}_i \rangle,$$

where $i = 1, \dots, 2g, j = 0, \dots, m-1$.
Let ρ be an irreducible $SU(2, \mathbb{C})$-metabelian representation of $K^{m,n}$.
Note that

$$\rho(\tau^j \tilde{x}_i) = \begin{pmatrix} \lambda_{ij} & 0 \\ 0 & \overline{\lambda_{ij}} \end{pmatrix} \quad \rho(\eta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

up to conjugation.

Sketch of proof(3/3)

(1)If m is odd, we can check directly that each $\lambda_{ij} = \pm 1$. Then the representation is reducible.

(2) If m is even, The anarogy of Lin's idea gives the number of irreducible metabelian representation is $\frac{|\Delta_K(-1)-1|}{2}$. □