Upsilon-invariants and Alexander polynomials of torus knots

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§1. Motivation and results

Ozsváth-Stipsicz-Szabó defined a concordant invariant $\Upsilon_K(t)$ (OSS'14/7).

- Torus knots
- Alternating knots
- Linearly independence of concordance group

A brief history of Υ after OSS.

- Torus knot formula in terms of semigroup (Borodzik and Livingston '14/8)
- Another reasonable definition (Livingston '15/1)
- Υ-invariant of L-space knot and Legendre transform (Borodzik-Hedden '15/5)
- g_4 of some connected-sum of torus knots, (Livingston-Van Cott '15/8)
- L-space knots in terms of formal semigroup (Feller-Krcatovich '16/2)

- (Infinite) iterated torus knots (not algebraic but L-space) (S.Wang '16/3)
- Whitehead doubles (OSS, Feller-J.Park-Ray '16/4)
- $\mathbb{Z}^\infty$-summand in $\mathcal{C}_\Delta$ (Kyungbae-M.H.Kim '16/4)
- Inequalities for general cable knots, Non-L-space cable knots (W.Chen '16/11)

Results

Let K be an L-space knot.

$$\Upsilon_{K_{p,q}}(t) = *^p \Upsilon_K(t) + \Upsilon_{T_{p,q}}(t)$$

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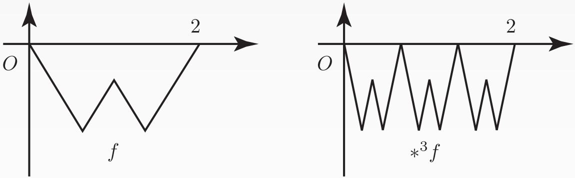
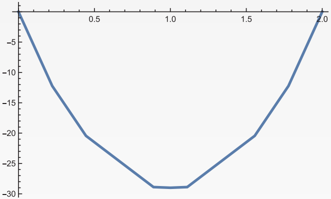


$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p)\Delta_{T_{p,q}}(t) \quad (c.f.)$$

$\Upsilon_{T(3,7)_{3,38}}$



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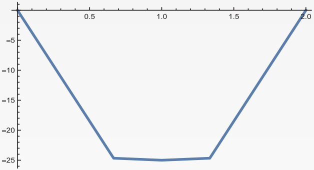
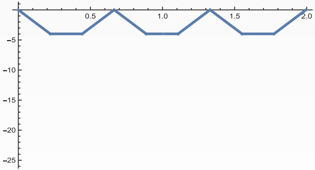


$= *^3\Upsilon_{T(3,7)}$



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We define integral

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Torus knot formula

$$I(T_{p,q}) = \int_0^2 \Upsilon_{T_{p,q}}(t)dt = -\frac{1}{3}(pq - \sum_{i=1}^{n} a_i)$$

$$\int_{S^1} \sigma_\omega(T_{p,q}) = -\frac{1}{3}\left(pq - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}\right)$$

§2. Definition

Definition 1 (L-space)

Y: $\mathbb{Q}HS^3$

Y is an L-space $\Leftrightarrow$ *for* $\mathfrak{s} \in Spin^c(Y)$

$$\widehat{HF}(Y, \mathfrak{s}) \cong \widehat{HF}(S^3)$$

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Definition 2 (L-space knot)

Let K be a knot in S^3.
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$\overset{def}{\Leftrightarrow} \exists\ n \in \mathbb{Z}_{>0}$ *s.t. the n-surgery is an L-space.*

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$$\{\text{Torus knots}\} \subset \{\text{Algebraic knots}\}$$
$$\subset \{\text{L-space iterated torus knots}\} \subset \{\text{L-space knots}\}$$
$$\subset \{\text{Strongly quasi-positive knots}\} \subset \{\text{quasi-positive knot}\}$$
$$= \{\text{transverse } \mathbb{C}\text{-link}\}$$

Definition 3 (Concordance)

Two knots K_0, K_1 are concordant
$\Leftrightarrow \exists$ *a smooth annulus embedding*

$$f : S^1 \times I \hookrightarrow S^3 \times I,$$

where $I = [0,1]$ *and* $f(S^1 \times i) = K_i$, *where* $i = 0, 1$.

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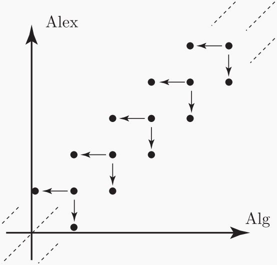
Furthermore this set admits group about the connected-sum.
This is called the *concordance group*.

Definition 4 (Knot Floer homology (Ozsváth-Szabó))

$C(K) := CFK^\infty(K)$

$CFK^\infty(K)$



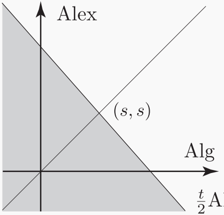
(trefoil)

A double complex with respect to a Heegaard decomposition of K.

Definition 5 (Υ-invariant (Ozsváth-Stipsitz-Szabó))

$$(C(K), \mathcal{F}_t)_s = \left\{ x \in C(K) \middle| \frac{t}{2} Alex(x) + \left(1 - \frac{t}{2}\right) Alg(x) \leq s \right\}$$

$(C(K), \mathcal{F}_t)_s$



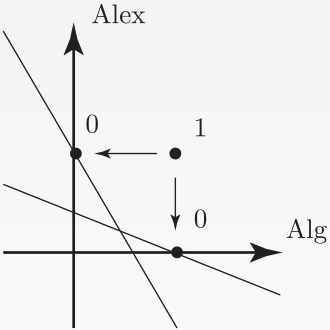
$$\nu(C(K), \mathcal{F}_t) = min\{s | H_0((C(K), \mathcal{F}_t)_s \to H_0(C) = \mathbb{F} \; surj\}$$

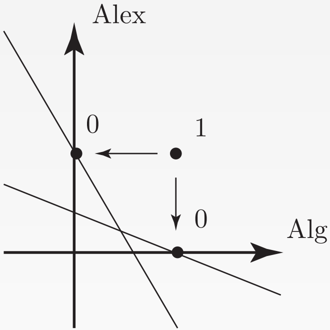
$$\Upsilon_K(t) = -2\nu(C(K), \mathcal{F}_t)$$

$$\Upsilon : \mathcal{C} \to C([0, 2])$$

$C([0, 2])$: the set of continuous functions.

Υ_K is a piece-wise linear function on $[0, 2]$.


Alex
0
1
0
Alg



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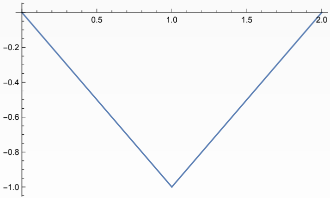

0.5
1.0
1.5
2.0
−0.2
−0.4
−0.6
−0.8
−1.0

Properties(OSS)

- $\Upsilon : \mathcal{C} \to C([0,2])$ (group homomorphism)
  1. $\Upsilon_{K^{mr}} = -\Upsilon_K$
  2. $\Upsilon_{K_1 \# K_2} = \Upsilon_{K_1} + \Upsilon_{K_2}$
- $\Upsilon(2-t) = \Upsilon(t)$
- $\Upsilon'_K(0) = -\tau(K)$
- $|\Upsilon_K(t)| \leq t g_4(K) \quad (0 < t < 1)$
- Let K be an alternating. $\Upsilon_K(t) = (1 - |t-1|)\frac{\sigma(K)}{2}$.

Definition 6 (Integral of Υ_K)

$$I : \mathcal{C} \to \mathbb{R} \quad I(K) = \int_0^2 \Upsilon_K(t)dt$$

K: an alternating.
$I(K) = \frac{\sigma(K)}{2} = -\tau(K)$

§3. Several formulas

Fact 7 (Torus knot formula (OSS))

Let K be an L-space knot.
$\Delta_K(t) = \sum_{k=0}^{n}(-1)^k t^{a_k}$ *(Alexander polynomial)*
$m_0 = 0, m_{2k} = m_{2k-1} - 1$
$m_{2k+1} = m_{2k} - 2(a_{2k} - a_{2k+1}) + 1$

$$\Upsilon_K(t) = \max_{0 \le 2i \le n}\{m_{2i} - ta_{2i}\}$$

$$a_0 > a_1 > \cdots > a_{2n}$$

$T(3,4)$

$$\Delta_{T(3,4)} = t^3 - t^2 + 1 - t^{-2} + t^{-3}$$

$$m_0 = 0, m_1 = -1, m_2 = -2, m_3 = -5, m_4 = -6$$

$$a_0 = 3, a_1 = 2, a_2 = 0, a_3 = -2, a_4 = -3$$

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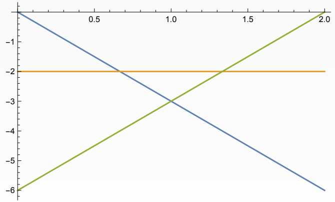

Formal semigroup

Fact 8 (Feller-Krcatovich and S.Wang)

Let K be an L-space knot.
$g := g(K)$ *Seifert genus*

$$\Upsilon_K(t) = -2 \min_{0 \le m \le 2g} \left\{ \#(S_K \cap [0, m)) + \frac{t(g-m)}{2} \right\}$$

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S_K :Formal semigroup.

$$\Delta_K(t) = \sum_{i=0}^{2n} (-1)^i t^{a_i}$$

$$(0 = a_0 < a_1 < a_2 < \cdots)$$

$$\frac{\Delta_K(t)}{1-t} = t^{s_0} + t^{s_1} + t^{s_2} + \cdots = \sum_{n=0}^{\infty} t^{s_n} \quad (0 = s_0 < s_1 < s_2 < \cdots)$$

$S_K = \{s_n | n \in \mathbb{Z}_{n \ge 0}\}$: Formal semigroup

Example($K = T_{3,7}$)

$\Delta_K(t) = 1 - t + t^3 - t^4 + t^6 - t^8 + t^9 - t^{11} + t^{12}$
$S_K = \{0, 3, 6, 7, 9, 10, 12\} \cup \mathbb{Z}_{n>12}$

$2 \notin S_K \Leftrightarrow 11 - 2 \in S_K$
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$s \in S_K \Leftrightarrow 11 - s \notin S_K$
$S_K = \langle 3, 7 \rangle_{\mathbb{Z}_{\geq 0}}$: semigroup generated by $3, 7$.
$S_{T_{p,q}} = \langle p, q \rangle_{\mathbb{Z}_{\geq 0}}$: semigroup generated by p, q.

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$$\Upsilon_K(t) = -2 \min_{0 \leq m \leq 2g} \left\{ \#(S_K \cap [0, m)) + \frac{t(g - m)}{2} \right\}$$

Formal semigroup of cable knots

Formal semigroup $S_{K_{p,q}}$

$p \geq 2$ and $q \geq p(2g(K) - 1)$, then

$$S_{K_{p,q}} = pS_K + q\mathbb{Z}_{\geq 0}.$$

For example:

$S_{T(2,3)_{3,5}} = 3\langle 2, 3\rangle_{\mathbb{Z}_{\geq 0}} + 5\mathbb{Z}_{\geq 0} = \langle 6, 9, 5\rangle_{\mathbb{Z}_{\geq 0}}$

Fact 9 (Torus knot relation (Feller and Krcatovich))

Let p, q be positive integers p, q (with relatively prime). Then, we have

$$\Upsilon_{T_{p,q+p}} = \Upsilon_{T_{p,q}} + \Upsilon_{T_{p,p+1}}$$

Torus knot formula

Let p, q be positive integers as above.

$$q/p = a_1 + \cfrac{1}{a_2 + \cdots + \frac{1}{a_n}} = [a_1, \cdots, a_n],$$

where a_i are non-negative integers.

Corollary 10 (Continued fraction expansion formula (FK))

$$\Upsilon_{T_{p,q}} = \sum_{i=1}^{n} a_i \Upsilon_{p_i, p_i+1},$$

where p_i is the denominator of $[a_i, \cdots, a_n]$

$\Upsilon(\mathcal{C}_{\text{torus}}) = \langle \Upsilon_{p,p+1} | p \in \mathbb{Z}_{p\geq 1} \rangle$
$\mathcal{C}_{\text{torus}}$: the subgroup generated by torus knots in $\mathcal{C}$.
$\{\Upsilon_{p,p+1} | p \in \mathbb{N}_{>1}\}$ are linearly independent in $C([0,2])$.

Corollary 11 (T.)

$$I(T_{p,q}) = -\frac{1}{3}(pq - \sum_{i=1}^{n} a_i)$$

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Proof

$$I(T_{p_i,p_i+1}) = -\frac{p_i^2 - 1}{3}$$

$$I(T_{p,q}) = \sum_{i=1}^{n} a_i I(T_{p_i,p_i+1}) = -\frac{1}{3}\sum_{i=1}^{n} a_i(p_i^2 - 1)$$

From the derivative at $t = 0$ of $\Upsilon_{T_{p,q}} = \sum_{i=1}^{n} a_i \Upsilon_{T_{p_i,p_i-1}}$ we have

$$(p-1)(q-1) = \sum_{i=1}^{n} a_i p_i (p_i - 1). \tag{1}$$

$$\sum_{i=1}^{n} a_i p_i = q + p - 1 \tag{2}$$

From (1), (2) we have $I(T_{p,q}) = -\frac{1}{3}(pq - \sum_{i=1}^{n} a_i)$. $\square$

Tristram-Levine signature

$\sigma_\omega(K) : S^1 \to \mathbb{R}$: Tristram-Levine signature.
S: the Seifert matrix

$$\sigma_\omega(K) = \text{sigature}((1-\omega)S + (1-\bar{\omega})S^T)$$

Fact 12

$$\int_{S^1} \sigma_\omega(T_{p,q}) = -\frac{1}{3}\left(pq - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq}\right)$$

$$(c.f.)\quad \tau(T_{p,q}) = \frac{1}{2}(pq - p - q + 1)$$

$$(c.f.)\quad I(T_{p,q}) = -\frac{1}{3}\left(pq - \sum_{i=1}^{n} a_i\right)$$

Cable knot formula

$K_{p,q}$: the (p,q)-cable knots, i.e., the satellite knot whose pattern is (p,q)-torus knot in the solid torus.

Fact 13

$$\Delta_{K_{p,q}} = \Delta_K(t^p)\Delta_{T_{p,q}}(t),$$

Fact 14 (Tristram-Levine signature formula of cable knots)

Let K be a knot.
Let p,q be coprime integers.

$$\sigma_{K_{p,q}}(\omega) = \sigma_K(\omega^p) + \sigma_{T_{p,q}}(\omega)$$

Fact 15 (τ-invariant formula of $K_{p,q}$ (Hom))

$$\tau(K_{p,q}) = \begin{cases} p\tau(K) + \frac{(p-1)(q-\epsilon)}{2} & \epsilon \neq 0 \\ \frac{(p-1)(q-1)}{2} & \epsilon = 0,\ q > 0 \\ \frac{(p-1)(q+1)}{2} & \epsilon = 0,\ q < 0 \end{cases}$$

In particular, if K is an L-space knot, then

$$\tau(K_{p,q}) = p\tau(K) + \tau(T_{p,q})$$

holds.

L-space cable knots

Fact 16 (Hedden, Hom)

Let p, q be positive integers with relatively prime.
$K_{p,q}$ is an L-space knot
$\Leftrightarrow$
K is an L-space knot and $q \geq p(2g(K) - 1)$.

§4. Main theorem

Theorem 17 (Upsilon invariant of L-space cable knots (T.))

Let K be an L-space knot.
Let p, q be coprime integers with $q \geq 2pg(K)$.

$$\Upsilon_{K_{p,q}} = *^p \Upsilon_K + \Upsilon_{T_{p,q}}.$$

where $*^p$ is the p-fold juxtaposition of the function.

Corollary 18 (T.)

Let K be an L-space knot. Then, $I(K_{p,q}) = I(K) + I(T_{p,q})$.

Proof.

$$\int_0^2 *^p \Upsilon_K dt = p \int_0^{\frac{2}{p}} *^p \Upsilon_K(t) dt$$

$$= p \int_0^2 *^p \Upsilon_K(\frac{s}{p})\frac{1}{p} ds = \int_0^2 \Upsilon_K(s) ds = I(K)$$

$$\int_0^2 \Upsilon_{K_{p,q}} dt = \int_0^2 *^p \Upsilon_K dt + \int_0^2 \Upsilon_{T_{p,q}} dt$$

$$\int_0^2 \Upsilon_K(t) dt + \int_0^2 \Upsilon_{T_{p,q}} dt = I(K) + I(T_{p,q}).$$

□

Integral values of L-space iterated torus knots

$$T(p_1, q_1; p_2, q_2; \cdots ; p_n, q_n) := (\cdots(T(p_1, q_1)_{p_2,q_2})_{\dots})_{p_n,q_n}$$

Theorem 19 ($I(K)$ of L-space iterated torus knots)

Let $T(p_1, q_1; p_2, q_2; \cdots ; p_n, q_n)$ be an L-space iterated torus knot.
$I(T(p_1, q_1; p_2, q_2; \cdots ; p_n, q_n)) = \sum_{i=1}^{n} I(T_{p_i,q_i})$

Proof.

$I(T(p_1, q_1; p_2, q_2; \cdots ; p_n, q_n))$
$= I(T(p_1, q_1; p_2, q_2; \cdots ; p_{n-1}, q_{n-1})) + I(T_{p_1,q_1})$
$= \cdots = \sum_{i=1}^{n} I(T_{p_i,q_i})$ □

Corollary 20

Let K, L be two L-space knots. Let p, q be coprime positive integers with $q \geq 2pg(K), \geq 2pg(L)$

$$\Upsilon_{K_{p,q}} = \Upsilon_{L_{p,q}} \Leftrightarrow \Upsilon_K = \Upsilon_L$$

Conjecture 21

If $\Upsilon_K(t) = \Upsilon_L(t)$, then $\Upsilon_{K_{p,q}}(t) = \Upsilon_{L_{p,q}}(t)$.

Conjecture 22 (General cabling Upsilon invariant formula)

$$\Upsilon_{K_{p,q}} = *^p \Upsilon_K + \Upsilon_{T_{p,q}}.$$

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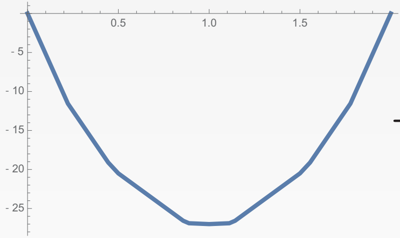
Counterexamples

This conjecture does not true in general.
In the case of $2g(K) - 1 \leq \frac{q}{p} < 2g(K)$ and $K_{p,q}$ is an L-space knot,
then $\Upsilon_{K_{p,q}}$ is obtained from the formal semigroup.
$S_{K_{p,q}} = pS_K + q\mathbb{Z}_{\geq 0}$.

$\Upsilon_{T(3,7)_{3,35}}$



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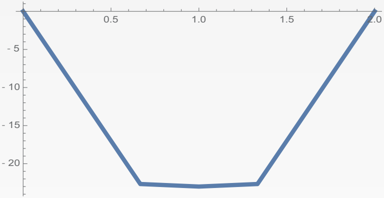
$\Upsilon_{T(3,35)}$



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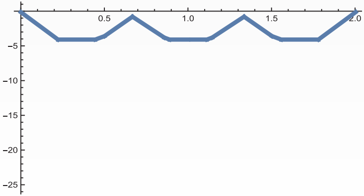
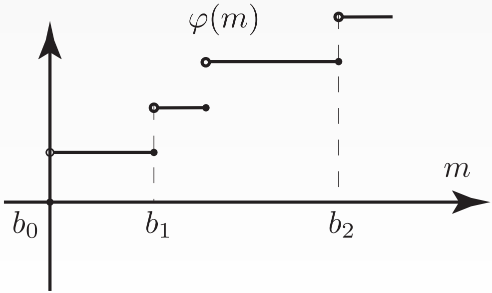


$\neq *^3\Upsilon_{T(3,7)}$

Proof.
Let K be an L-space.
Let S_K be a formal semigroup.

$$E_k(t) = \begin{cases} 0 & t \leq k \\ 1 & t > k \end{cases}$$

$$\varphi(m) = \#(S_K \cap [0, m)) = \sum_{s \in S_K} E_s(m)$$



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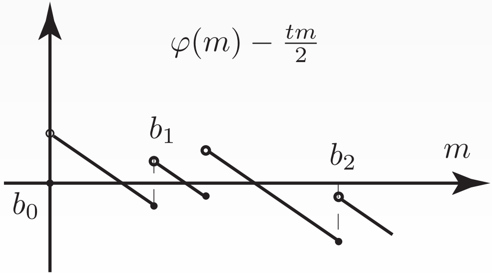
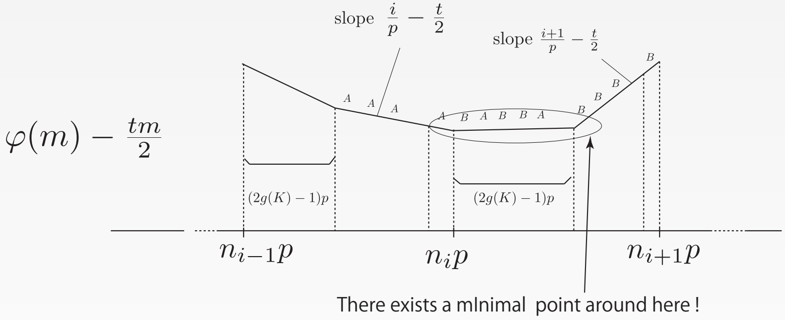
$S_K = \{0, 3, 6, 7, 9, 10, 12\} \cup \mathbb{Z}_{>10}$.
$S_K = \{0 =$
$b_0, b_0+1, \cdots, b_0+n_0-1, b_1, b_1+1, \cdots, b_1+n_1-1, b_2, b_2+1, \cdots\}$.
Namely, we have $\hat{S}_K = \{0 = b_0, b_1, b_2, \cdots b_k = 2g\}$
$c_i = b_{i+1} - (b_i + n_i)$.
$\hat{S}_K = \{0, 3, 6, 9, 12\}$, $k = \#\hat{S}_K = 5$.

$$\min\left\{\sum_{s\in S_K} E_s(m) - \frac{t}{2}m\right\} = \min\left\{\sum_{i=0}^{l} n_i - \frac{t}{2}\sum_{i=0}^{l-1} c_i | l \in 1, 2, \cdots, k\right\}$$

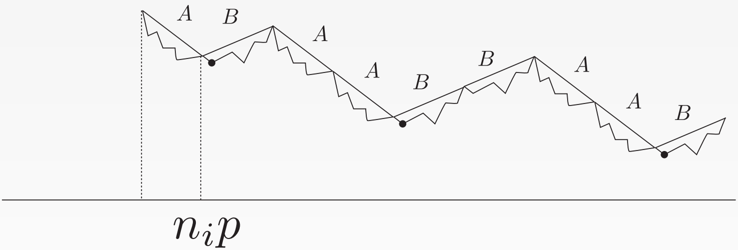


If $\frac{2i}{p} \leq t < \frac{2(i+1)}{p}$, then we consider the function $\varphi(m) - \frac{tm}{2}$

$$\varphi(m) - \frac{tm}{2} = \#(S_{K_{p,q}} \cap [0, m)) - \frac{tm}{2}$$



$A : \sum_{s \in S_A} E_s(m) - \frac{tm}{2}$
where $S_A = \{qj \mod p | j = 0, 1, 2 \cdots, i-1\} \subset \{0, 1, 2, \cdots, p-1\}$
$B : \sum_{s \in S_B} E_s(m) - \frac{tm}{2}$
where $S_B = \{qj \mod p | j = 0, 1, 2 \cdots, i\} \subset \{0, 1, 2, \cdots, p-1\}$


A
B
A
A
B
B
A
A
B
$n_i p$