トーラス束と準トーラス束上の Sol 構造の具体的構成と対合の分類

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It is well-known that any torus bundle over the circle with Anosov monodromy and any torus semi-bundle (Sapphire space), which has a double cover homeomorphic to such a torus bundle, admit Sol structure.

In this talk, I give an explicit construction of Sol structure of these manifolds, and then give a progress report on my project towards determination of their isometry groups and classification of their involutions.

In fact, we describe an algorithm to determine the isometry groups and all involutions of such torus semi-bundles, through an explicit example.

Plan

(1) Definitions of torus bundles and torus semi-bundles
(2) Sol geometry
(3) Sol structure on torus bundles
(4) Sol structure on torus semi-bundles
(5) Example

(1)Definitions of torus bundles and torus semi-bundles

Let $A \in SL(2,\mathbb{Z})$.

$$\begin{array}{ccccccc} & & \mathbb{R}^2 & \xrightarrow[\cong]{A} & \mathbb{R}^2 & & \\ & & \downarrow & & \downarrow & & \\ T^2 & = & \mathbb{R}^2/\mathbb{Z}^2 & \xrightarrow[\cong]{A} & T^2 & = & \mathbb{R}^2/\mathbb{Z}^2 \end{array}$$

$$A : T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2; [x] \mapsto [Ax]$$

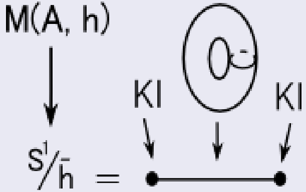
Definition (Torus bundle)

$M_A := T^2 \times \mathbb{R}/(x,t) \sim (Ax, t+1)$ is called the torus bundle over the circle with monodromy A.

Definition (Torus semi-bundle)

Let h be a free involution ($i.e.$ $h^2 = id_{M_A}$, $\mathrm{Fix}(h) = \emptyset$) on the torus bundle which satisfies the following condition.

$$\begin{array}{ccc} M_A & \xrightarrow{\;h\;} & M_A \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{\;\bar{h}\;} & S^1 \\ z & \longrightarrow & \bar{z} \end{array}$$



Then, $M(A,h) := M_A/h$ is called a torus semi-bundle.

ultimate goal $(*)$

To classify involutions of torus semi-bundles

Previous studies

- Sakuma [1985] gave a classification of involutions on torus bundles.
- Barreto-Goncalves-Vendruscolo [2016] gave a classification of free involutions on torus semi-bundles.

Definition

Let r and r' be involutions on a 3-manifold M.
Then r and r' are equivalent
$\Leftrightarrow \exists f : M \to M$ homeo. s.t. $f \circ r \circ f^{-1} = r'$

$$\begin{array}{ccc} M & \xrightarrow[\cong]{f} & M \\ r \downarrow & \circlearrowleft & \downarrow r' \\ M & \xrightarrow[\cong]{f} & M \end{array}$$

If $|\mathrm{Tr}A| \geq 3$, then M_A and $M(A,h)$ admit Sol structure.
By the orbifold theorem, $(*)$ is equivalent to the classification of the order two elements of $\mathrm{Isom}(M(A,h))$.

Theorem (Sakuma(1985))

(1)If M_A is an orientable torus bundle, then $1 \leq |\mathrm{Inv}(M_A)| \leq 21$.
(2)If M_A is a non-orientable torus bundle, then $1 \leq |\mathrm{Inv}(M_A)| \leq 7$.

Here $\mathrm{Inv}(M)$ denotes the set of all equivalence classes of involutions on M, and $|S|$ denotes the cardinality of S.

Example (Sakuma(1985))

For the torus bundle M_A with $A = \begin{pmatrix} 89 & 20 \\ 40 & 9 \end{pmatrix}$, $|\mathrm{Inv}(M_A)| = 21$.

Theorem (F)

There is an algorithm to determine the set $\mathrm{Inv}(M(A, h))$.

Example

For $A = \begin{pmatrix} 89 & 20 \\ 40 & 9 \end{pmatrix}$, $|\mathrm{Inv}(M(A, h))| = 10$.

(2) Sol geometry

The Lie group Sol is defined to be the semi direct product $\mathbb{R}^2 \rtimes_T \mathbb{R}$.

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow \mathrm{Sol} \underset{\iota}{\rightleftarrows} \mathbb{R} \longrightarrow 1$$

Here,

$$T : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$$
$$z \mapsto \begin{pmatrix} e^{-z} & 0 \\ 0 & e^{z} \end{pmatrix}$$

If we identify Sol with $\mathbb{R}^2 \times \mathbb{R}$ by

$$\mathbb{R}^2 \times \mathbb{R} \to \mathrm{Sol}$$
$$(x, z) \mapsto x\iota(z),$$

then the multiplication is given by

$$(\boldsymbol{x}, z)(\boldsymbol{x}', z') = (\boldsymbol{x} + T(z)\boldsymbol{x}',\ z + z').$$

With respect to a natural Riemann metric on Sol, we have

$$1 \longrightarrow \mathrm{Sol} \longrightarrow \mathrm{Isom}(\mathrm{Sol}) \rightleftarrows \mathrm{D}(4) \longrightarrow 1$$

Here $\mathrm{D}(4)$ is the dihedral group which consists of 8 maps of $\mathbb{R}^3$ given by $(x, y, z) \mapsto (\pm x, \pm y, z)$ and $(x, y, z) \mapsto (\pm y, \pm x, -z)$.

Definition

A 3-manifold M admits Sol structure
$\Leftrightarrow \exists \Gamma < \mathrm{Isom}(\mathrm{Sol})$; discrete torsion-free group s.t. $M \cong \mathrm{Sol}/\Gamma$

This is equivalent to the existence of the developing map $D : \widetilde{M} \to \mathrm{Sol}$, where $\widetilde{M}$ is the universal covering of M, and the holonomy homo $\rho : \pi_1(M) \to \mathrm{Isom}(\mathrm{Sol})$.
s.t.

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow[\cong]{D} & \mathrm{Sol} \\ {\scriptstyle g}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \rho(g)} \\ \widetilde{M} & \xrightarrow[\cong]{D} & \mathrm{Sol} \end{array} \qquad \forall g \in \pi_1(M)$$

- **The structure of $\pi_1(M_A)$**

$$1 \longrightarrow \pi_1(T^2) \longrightarrow \pi_1(M_A) \rightleftarrows \langle \gamma \rangle \longrightarrow 1, \qquad \pi_1(T^2) = \mathbb{Z}^2$$

$$\pi_1(M_A) = \langle\ \mathbb{Z}^2, \gamma \mid \gamma \boldsymbol{n} \gamma^{-1} = A\boldsymbol{n},\ \boldsymbol{n} \in \mathbb{Z}^2\ \rangle$$

- **The universal cover $\widetilde{M_A}$ and the action of $\pi_1(M_A)$**

$$\widetilde{M_A} \cong \widetilde{T^2} \times \mathbb{R} = \mathbb{R}^2 \times \mathbb{R}$$

The action of $\pi_1(M_A)$ on $\widetilde{M_A} = \mathbb{R}^2 \times \mathbb{R}$ is given by

$$\begin{aligned} \boldsymbol{n} \cdot (\boldsymbol{x}, t) &= (\boldsymbol{x} + \boldsymbol{n}, t) \qquad (\boldsymbol{n} \in \mathbb{Z}^2) \\ \gamma \cdot (\boldsymbol{x}, t) &= (A\boldsymbol{x}, t + 1). \end{aligned}$$

(3) Sol structure on torus bundles

Recall that $A \in SL(2,\mathbb{Z}), |\mathrm{Tr}A| \geq 3$.

<u>Case 1.</u> $\mathrm{Tr}A \geq 3$

Choose $Q \in GL^+(2,\mathbb{R})$, s.t. $QAQ^{-1} = \begin{pmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{pmatrix}$.

We define the following developing map D and holonomy homo ρ:

$$\begin{aligned} D : \widetilde{M_A} &\to \mathrm{Sol} \\ (x,t) &\mapsto (Qx, t\tau) \end{aligned}$$

$$\begin{aligned} \rho : \pi_1(M_A) &\to \mathrm{Sol} < \mathrm{Isom}(\mathrm{Sol}) \\ \boldsymbol{n} &\mapsto Q\boldsymbol{n} \in \mathbb{R}^2 \qquad (\boldsymbol{n} \in \mathbb{Z}^2) \\ \gamma &\mapsto \tau = \iota(\tau) \end{aligned}$$

$$1 \longrightarrow \mathbb{R}^2 \longrightarrow \mathrm{Sol} \underset{\iota}{\rightleftarrows} \mathbb{R} \longrightarrow 1$$

$\pi_1(M_A) = \mathbb{Z}^2 \rtimes \langle \gamma \rangle$

Then

$$
\begin{array}{ccc}
\widetilde{M_A} & \xrightarrow[\cong]{D} & \mathrm{Sol} \\
g \big\downarrow & \circlearrowleft & \big\downarrow \rho(g) \\
\widetilde{M_A} & \xrightarrow[\cong]{D} & \mathrm{Sol}
\end{array}
\qquad \forall g \in \pi_1(M_A) = \mathbb{Z}^2 \rtimes \langle \gamma \rangle
$$

Put $\Gamma := \rho(\pi_1(M_A))$.
Then we obtain $M_A = \widetilde{M_A}/\pi_1(M_A) \cong \mathrm{Sol}/\Gamma$.

Case 2. $\mathrm{Tr}A \leq -3$

Choose $Q \in GL^{+}(2,\mathbb{R})$, s.t. $QAQ^{-1} = \begin{pmatrix} -e^{-\tau} & 0 \\ 0 & -e^{\tau} \end{pmatrix}$.

We use the developing map D and define the following holonomy homo ρ':

$$D : \widetilde{M}_A \to \mathrm{Sol} \qquad\qquad \rho' : \pi_1(M_A) \to \mathrm{Isom}(\mathrm{Sol})$$

$$(x,t) \mapsto (Qx, t\tau) \qquad\qquad \boldsymbol{n} \mapsto Q\boldsymbol{n} \in \mathbb{R}^2 \quad (\boldsymbol{n} \in \mathbb{Z}^2)$$

$$\gamma \mapsto f\tau$$

where $f : \mathrm{Sol} \to \mathrm{Sol}; (x,t) \mapsto (-x,t)$.

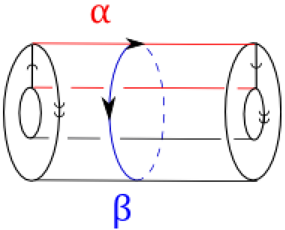
Then

$$\begin{array}{ccc} \widetilde{M}_A & \xrightarrow[\cong]{D} & \mathrm{Sol} \\ g\downarrow & \circlearrowleft & \downarrow\rho'(g) \\ \widetilde{M}_A & \xrightarrow[\cong]{D} & \mathrm{Sol} \end{array} \qquad \forall g \in \pi_1(M_A)$$

Put $\Gamma := \rho'(\pi_1(M_A))$. Then we obtain $M_A = \widetilde{M}_A/\pi_1(M_A) \cong \mathrm{Sol}/\Gamma$.

(4)Sol structure on torus semi-bundles

The torus semi-bundle $M(A,h)$ is the union of two copies of twisted I-bundle KI over the Klein bottle glued by a homeomorphism, B, of $\partial KI \cong T^2$.



Here,
$$B : \pi_1(\partial KI) \to \pi_1(\partial KI)$$
$$(\alpha, \beta) \mapsto (\alpha, \beta)B$$

So we put $N_B := KI \underset{B}{\cup} KI = M(A,h)$, where $B \in SL(2,\mathbb{Z})$.

$$KI = [0,1] \times S^1 \times [-1,1]/(0,\theta,s) \sim (1,-\theta,-s)$$

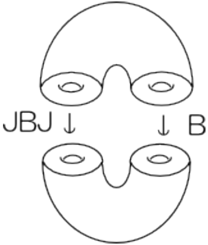
Observation

Let $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A = JB^{-1}JB.$
Then the torus semi bundle N_B has the torus bundle M_A as a double covering.

(Proof)
Note that KI has the double covering $S^1 \times S^1 \times [-1, 1]$.

Its covering transformation group is generated by

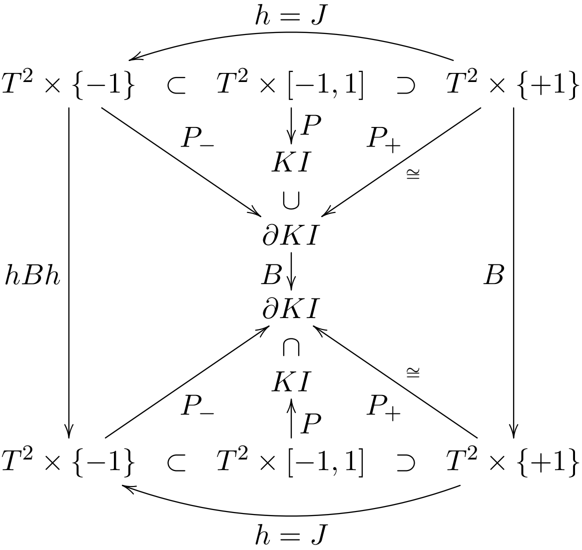
$$h : S^1 \times S^1 \times [-1, 1] \to S^1 \times S^1 \times [-1, 1]$$
$$(x, y, t) \mapsto (x + 1/2, -y, -t).$$



The double covering of N_B is the union of two copies of $S^1 \times S^1 \times [-1, 1]$ glued by a homeomorphism of $\partial(S^1 \times S^1 \times [-1, 1]) = T^2 \times \{-1, 1\}$.

$$
\begin{array}{ccccc}
T^2\times\{-1\} & \subset & T^2\times[-1,1] & \supset & T^2\times\{+1\} \\
 & \searrow^{P_-} & \downarrow P & {}^{P_+}\swarrow \cong & \\
 & & KI & & \\
 & & \cup & & \\
 & & \partial KI & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \partial KI & & \\
 & & \cap & & \\
 & & KI & & \\
 & \nearrow^{P_-} & \uparrow P & {}^{P_+}\nwarrow \cong & \\
T^2\times\{-1\} & \subset & T^2\times[-1,1] & \supset & T^2\times\{+1\}
\end{array}
$$



From this picture, the double covering $\widehat{N}_B$ is identified with M_A, where $A = (hBh)^{-1}B = hB^{-1}hB = JB^{-1}JB$.

Remark

If $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, then $A = JB^{-1}JB = \begin{pmatrix} 1+2bc & 2bd \\ 2ac & 1+2bc \end{pmatrix}$.

Hence, $M_A = \widehat{N}_B$ admits Sol structure
$\Leftrightarrow |\mathrm{Tr} A| = |2(1+2bc)| \geq 3 \Leftrightarrow bc \neq 0, -1.$

In the remainder, we assume $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $bc \neq 0, -1$.

<u>Recall</u>

$M_A = \widetilde{M}_A / \pi_1(M_A) \cong \mathrm{Sol}/\Gamma$, where $\Gamma := \rho(\pi_1(M_A))$.

$$\begin{array}{ccc} \widetilde{M}_A & \xrightarrow[\cong]{D} & \mathrm{Sol} \\ {\scriptstyle g \in \pi_1(M_A)} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \rho(g)} \\ \widetilde{M}_A & \xrightarrow[\cong]{D} & \mathrm{Sol} \end{array}$$

The developing homeo D maps (x,t) to $(Qx, t\tau)$,
where $Q \in GL^+(2,\mathbb{R})$, s.t. $QAQ^{-1} = \begin{pmatrix} \pm e^{-\tau} & 0 \\ 0 & \pm e^{\tau} \end{pmatrix}$.

Let N_B be a torus semi-bundle.
For simplicity, we consider the case that $B \equiv I \in SL(2, \mathbb{Z}/2\mathbb{Z})$.

Let $r : M_A \to M_A$ be the covering involution of the double covering $M_A \to N_B$.

By using the fact that

$$JAJ = J(JB^{-1}JB)J = B^{-1}JBJ = A^{-1},$$

the involution r lifts to the following involution.

$$r : \mathbb{R}^2 \times \mathbb{R} = \widetilde{M_A} \to \widetilde{M_A} = \mathbb{R}^2 \times \mathbb{R}$$
$$(\boldsymbol{x}, t) \mapsto (J\boldsymbol{x} + \frac{1}{2}\,\boldsymbol{e}_1, -t),$$

where $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

In order to put a Sol structure on N_B, we have only to show that we can choose the developing map $D : \widetilde{M}_A \to \mathrm{Sol}$ so that $DrD^{-1} \in \mathrm{Isom}(\mathrm{Sol})$.

Note that

$$DrD^{-1} : (\boldsymbol{x}, z) \mapsto (QJQ^{-1}\boldsymbol{x} + \frac{1}{2}\, Q\boldsymbol{e}_1, -z)$$

$$\begin{array}{ccc} \widetilde{M}_A & \xrightarrow[\cong]{D} & \mathrm{Sol} \\ r \downarrow & & \vdots\, DrD^{-1} \\ \widetilde{M}_A & \xrightarrow[\cong]{D} & \mathrm{Sol} \end{array}$$

Note also that

$$\begin{aligned} QJQ^{-1}\begin{pmatrix} \pm e^{-\tau} & 0 \\ 0 & \pm e^{\tau} \end{pmatrix}(QJQ^{-1})^{-1} &= QJ(Q^{-1}\begin{pmatrix} \pm e^{-\tau} & 0 \\ 0 & \pm e^{\tau} \end{pmatrix}Q)JQ^{-1} \\ &= QJAJQ^{-1} \\ &= QA^{-1}Q^{-1} = \begin{pmatrix} \pm e^{-\tau} & 0 \\ 0 & \pm e^{\tau} \end{pmatrix}^{-1}. \end{aligned}$$

Hence $QJQ^{-1} = \pm \begin{pmatrix} 0 & 1/k \\ k & 0 \end{pmatrix} \ (k > 0)$.

Let $Q' = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} Q$.

Then we have

$$\begin{aligned} Q'JQ'^{-1} &= \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} QJQ^{-1} \begin{pmatrix} 1/k & 0 \\ 0 & 1 \end{pmatrix} \\ &= \pm \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1/k \\ k & 0 \end{pmatrix} \begin{pmatrix} 1/k & 0 \\ 0 & 1 \end{pmatrix} \\ &= \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

and

$$Q'AQ'^{-1} = \begin{pmatrix} \pm e^{-\tau} & 0 \\ 0 & \pm e^{\tau} \end{pmatrix}.$$

The map $(\boldsymbol{x}, z) \mapsto \left(\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x}, -z\right)$ is the element of $\mathrm{D}(4) < \mathrm{Isom}(\mathrm{Sol})$.

Recall

$$1 \longrightarrow \mathrm{Sol} \longrightarrow \mathrm{Isom}(\mathrm{Sol}) \rightleftarrows \mathrm{D}(4) \longrightarrow 1$$

$$\mathrm{D}(4) = \{(x, y, z) \mapsto (\pm x, \pm y, z), (x, y, z) \mapsto (\pm y, \pm x, -z)\}$$

Thus we have $DrD^{-1} \in \mathrm{Isom}(\mathrm{Sol})$ and $M_A / \langle DrD^{-1} \rangle = N_B$, where $M_A = \mathrm{Sol}/\Gamma$. Hence N_B admits Sol structure.

(5)Example

For $A = \begin{pmatrix} 89 & 20 \\ 40 & 9 \end{pmatrix}$, $|\mathrm{Inv}(M_A)| = 21$. (Sakuma [1985])

Let $B = \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}$, then M_A is the double covering of N_B.
The torus semi-bundle N_B admits 3 distinct classes of free involutions. (Barreto-Goncalves-Vendruscolo [2016])

Example $|\mathrm{Inv}(N_B)| = 10$

- Step 1 Determine $\mathrm{Isom}(M_A)$
- Step 2 Choose $r \in \mathrm{Isom}(M_A)$ s.t. $M_A/\langle r \rangle = N_B$
- Step 3 Determine $\mathrm{Isom}(N_B)$ and classify the order 2 elements of it up to conjugacy

Step 1. (Determine $\mathrm{Isom}(M_A)$)

Replace A as $A = XAX^{-1} = \begin{pmatrix} 49 & 20 \\ 120 & 49 \end{pmatrix}$. $X := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in GL(2, \mathbb{Z})$

Let $B = \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}$, then $JB^{-1}JB = A$.

Note $JAJ = A^{-1}$, $A = B^2 (\because JB^{-1}J = B)$

$$\tau_0 : \widetilde{M_A} \to \widetilde{M_A} \quad (\boldsymbol{x}, t) \mapsto (B\boldsymbol{x}, t + \frac{1}{2})$$

$$J : \widetilde{M_A} \to \widetilde{M_A} \quad (\boldsymbol{x}, t) \mapsto (J\boldsymbol{x}, -t)$$

$$-E : \widetilde{M_A} \to \widetilde{M_A} \quad (\boldsymbol{x}, t) \mapsto (-\boldsymbol{x}, t)$$

$$1 \to \mathrm{Coker}(A - E) \to \mathrm{Isom}_0(M_A) \rightleftarrows \langle \tau_0 \mid {\tau_0}^2 = 1 \rangle \to 1$$

$$1 \longrightarrow \mathrm{Isom}_0(M_A) \longrightarrow \mathrm{Isom}(M_A) \rightleftarrows \langle J \rangle \oplus \langle -E \rangle \longrightarrow 1 \qquad J^2 = (-E)^2 = 1$$

$${\tau_0} \boldsymbol{v} {\tau_0}^{-1} = B\boldsymbol{v} \quad (\boldsymbol{v} \in \mathbb{Z}^2)$$

$$J\boldsymbol{v}J = -AJ\boldsymbol{v} \qquad J\tau_0 J = {\tau_0}^{-1}$$

$$(-E)\boldsymbol{v}(-E) = -\boldsymbol{v} \qquad (-E)\tau_0(-E) = \tau_0$$

Step 2. (Choose $r \in \mathrm{Isom}(M_A)$ s.t. $M_A/\langle r \rangle = N_B$)

$\mathrm{Coker}(A - E) = \mathbb{Z}_4 \oplus \mathbb{Z}_{24}$
$(\bar{1}, \bar{0}) = \boldsymbol{e}_1, (\bar{0}, \bar{1}) = \boldsymbol{e}_2$

An isometry $(\bar{0}, \bar{12})J$ is a free involution of M_A and it is possible to check $M_A/\langle(\bar{0}, \bar{12})J\rangle = N_B$.

Step 3. (Determine $\mathrm{Isom}(N_B)$ and classify the order 2 elements of it up to conjugacy)

$$1 \to (\mathbb{Z}_2 \oplus \mathbb{Z}_{24}) \rtimes \langle \tau_0 \mid {\tau_0}^2 = 1 \rangle \to \mathrm{Isom}(N_B) \rightleftarrows \langle -E \mid (-E)^2 = 1 \rangle \to 1$$

$$\mathrm{Isom}(N_B) = \left\langle f_1, f_2, \tau_0, -E \;\middle|\; \begin{array}{l} {f_1}^2 = {f_2}^{24} = [f_1, f_2] = {\tau_0}^2 = (-E)^2 = 1, \\ \tau_0 f_1 \tau_0 = f_1,\ \tau_0 f_2 \tau_0 = f_1 {f_2}^5, \\ (-E) f_1 (-E) = f_1, (-E) f_2 (-E) = {f_2}^{-1}, \\ (-E) \tau_0 (-E) = \tau_0 \end{array} \right\rangle$$

$f_1 = (\bar{2}, \bar{0}), f_2 = (\bar{0}, \bar{1})$

Thank you for your attention.