Homology 4-balls with complexity zero and collapsing of shadows

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Introduction (1/4)

※ In this talk we assume that manifolds are compact, oriented and smooth otherwise mentioned.

Definition (exotic)

Two manifolds X and Y are said to be **exotic** if they are homeomorphic but NOT diffeomorphic.

- Milnor found manifolds exotic to S^7 in 1956.
  There are 28 different smooth structures on S^7.
- Any 3-manifold has no exotic pairs.
- Euclidean space $\mathbb{R}^n$ has an exotic pair iff $n = 4$.
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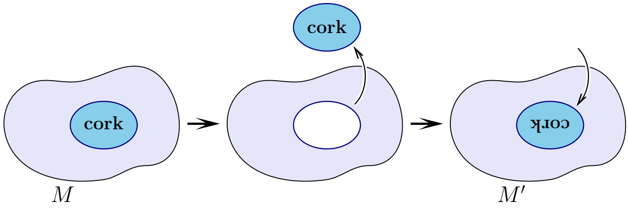
Introduction (2/4)

Theorem (Matveyev, Curtis-Freedman-Hsiang-Stong '96)

Let (M, M') be an exotic pair of simply-connected closed 4-manifolds. Then there exists a contractible submanifold C of M s.t. M' is obtained from M by a surgery along C.

Such a contractible submanifold C is called a **cork**.



Motivation

Classification and characterization of corks.

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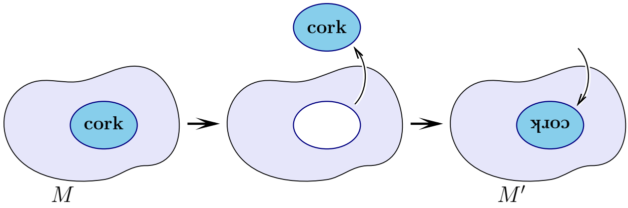

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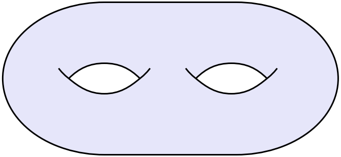
Introduction (3/4)

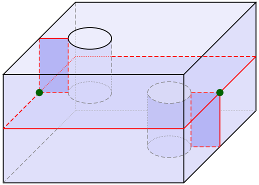
Description of 4-manifolds:

- **Kirby diagram** ··· link + framings
- **shadow** ··· 2-dimensional (simple) polyhedron + gleams

Example (simple polyhedra)



compact surface



Bing's house

Introduction (3/4)

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- **Kirby diagram** $\cdots$ link + framings
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Definition (shadow-complexity sc)

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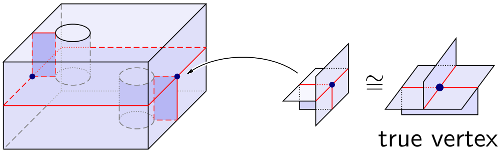


Introduction (4/4)

Main theorem (N.)

M is an integral homology 4-ball w/ $sc(M) = 0 \iff M \cong D^4$.

Corollary (N.)

There are no corks w/ $sc = 0$.

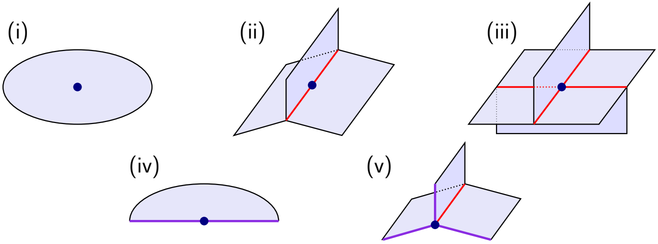
Remark.

- There are (infinitely) many rational homology 4-balls w/ $sc = 0$.
- There are (infinitely) many corks w/ $sc = 1$ (N. '15).

The plan of this talk

§1 Preliminaries - Simple polyhedron (1/2)

- **simple polyhedron**: a compact topological space X s.t. each point of X has a neighborhood homeo. to one of:

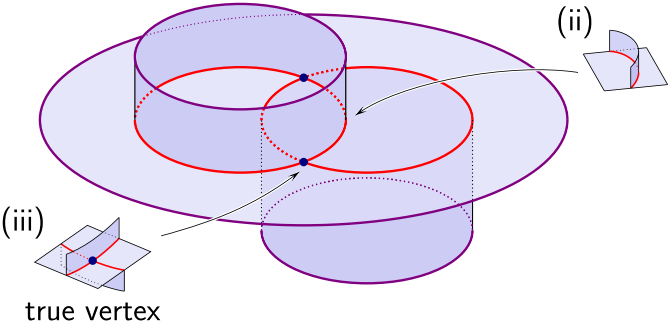


- singular set: set of points of type (ii), (iii) and (v).
- **true vertex**: a point of type (iii).
- region: a conn. component of $X \setminus Sing(X)$.
- boundary: set of points of type (iv) and (v).

§1 Preliminaries - Simple polyhedron (2/2)

Example (simple polyhedron)

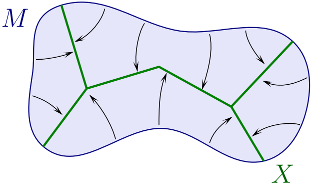
$D^2 \cup$ (two annuli):



§1 Preliminaries - Shadow (1/3)

Definition (shadow)

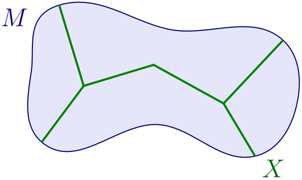
Let M be a 4-manifold w/ boundary and $X \subset M$ be a simple polyhedron.
If $M \searrow X$, and if X is proper and locally flat in M,
then X is called a **shadow** of M.



Examples :

- D^2-bundle over Σ has a shadow Σ.
- Let $M \to D^2$ be a Lefschetz fibration whose regular fiber is F w/ vanishing cycles $\gamma_1, \dots, \gamma_n$. Then a simple polyhedron obtained from F by attaching n 2-disks along $\gamma_1, \dots, \gamma_n$ is a shadow of M.

§1 Preliminaries - Shadow (2/3)



Studies of shadows:

- Spin^c structure
- (almost) complex structure
- Stein structure

} $\cdots$ 4-manifolds

- hyperbolic structure (volume)
- stable maps into $\mathbb{R}^2$

} $\cdots$ 3-manifolds

The boundary ∂X is a link (or a trivalent graph) in the 3-manifold ∂M.

- (quantum) invariants

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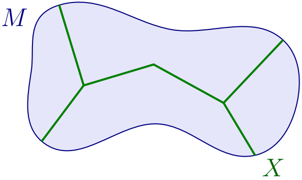

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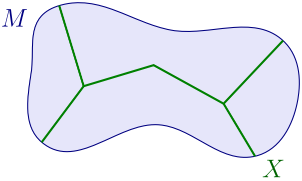

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Definition (shadow-complexity sc)

The **shadow-complexity** $sc(M)$ of a 4-manifold M is defined as the minimum number of true vertices of a shadow of M.

Theorem (Costantino '06)

M is a closed 4-manifold w/ $sc^{sp}(M) = 0$
$\iff M \cong S^4, S^2 \times S^2, \mathbb{CP}^2, \overline{\mathbb{CP}^2}, \mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ *or* $\overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2}$.

Theorem (Martelli '11)

M is a closed 4-manifold w/ $sc(M) = 0$
$\iff M \cong M' \#_k \mathbb{CP}^2$, *where M' is a "4-dimensional graph manifold".*

Thus the closed 4-manifolds w/ $sc = 0$ have been completely classified.
Our main result is the case of the 4-manifolds with boundary.

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Theorem A (N.)

Let X be a shadow of M^4, and let $Y \subset X$ be a simple polyhedron. If $X \searrow Y$, then Y is a shadow of a 4-manifold diffeo. to M.

Theorem B (N.)

Every acyclic simple polyhedron w/ no true vertices collapses onto D^2.

Remark.

- A PL-manifold has a unique smoothing in $\dim \leq 6$.
- X is said to be acyclic if $H_*(X;\mathbb{Z}) \cong H_*(\{\mathrm{pt.}\};\mathbb{Z})$.
- There are 3 types of collapsing in Theorem B.

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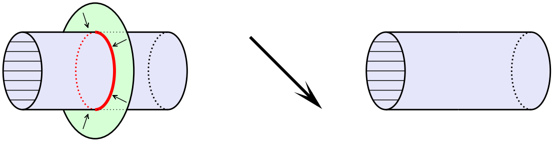
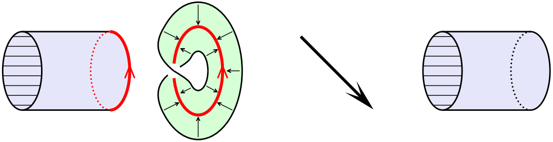


collapse (a)



collapse (b)

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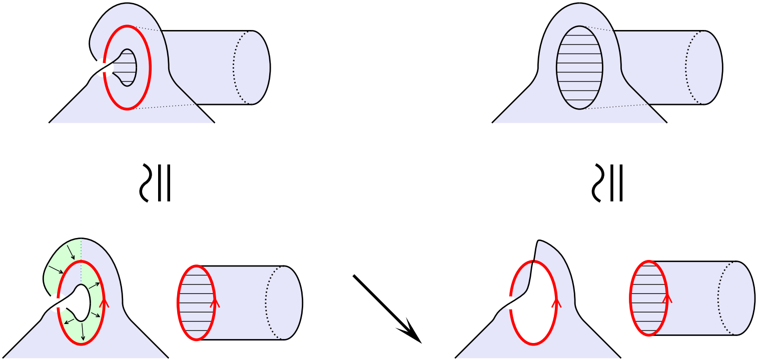

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Theorem A (N.) (recall)

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Theorem B (N.) (recall)

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The main theorem is a consequence of Theorem A and B.

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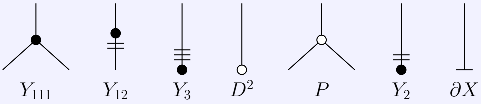
Every simple polyhedron X whose singular set consists of circles can be represented by a graph w/ vertices of the following:



Theorem B (N.) (recall)

Every acyclic simple polyhedron w/ no true vertices collapses onto D^2.

We used Martelli's graph to prove Theorem B.

- If X is acyclic, then its graph is a tree.
- Introduce some moves of graphs which corresponds to collapsing.
- etc...

A graph of X is a tree $\not\Rightarrow$ X is acyclic.

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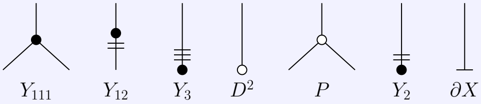

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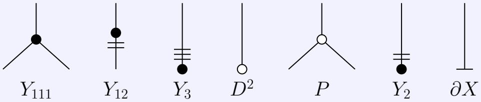

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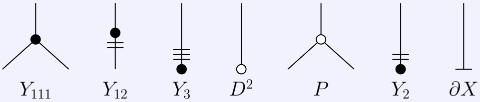

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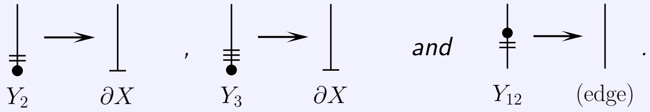
Proposition (N.)

Suppose that a graph G of a simple polyhedron X is a tree. Then the following moves of G preserve the rational homology of X.



This immediately yields the following:

Proposition (N.)

X is a shadow of $\mathbb{Q}\mathrm{HB}^4$ w/o true vertices.
$\Longleftrightarrow$ A graph of X is a tree s.t. the resulting graph after applying the above moves encodes an acyclic polyhedron.

Summary

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