An obstruction to the existence of embeddings between right-angled Artin groups

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Nihon University, December 21, 2016

Right-angled Artin groups

Γ: a finite (simplicial) graph
$V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$: the vertex set of Γ
$E(\Gamma)$: the edge set of Γ

Definition

The right-angled Artin group (RAAG) $G(\Gamma)$ on Γ is the group given by the following presentation:

$$G(\Gamma) = \langle\, v_1, v_2, \ldots, v_n \mid [v_i, v_j] = 1 \text{ } if \text{ } \{v_i, v_j\} \notin E(\Gamma) \,\rangle.$$

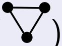
$G(\Gamma_1) \cong G(\Gamma_2)$ if and only if $\Gamma_1 \cong \Gamma_2$.

Example

$G($  $) \cong \mathbb{Z}^3$

$G($  $) \cong \mathbb{Z} \times F_2$

$G($  $) \cong \mathbb{Z}^2 * \mathbb{Z}$

$G($  $) \cong F_3$

Note: $G(\Gamma) = \langle\ v_1, v_2, \ldots, v_n \ \mid\ [v_i, v_j] = 1 \ \mathit{if}\ \{v_i, v_j\} \notin E(\Gamma)\ \rangle$

Motivation and main results

Problem (Crisp-Sageev-Sapir, 2008)

For given two finite graphs Λ and Γ, decide whether $G(\Lambda)$ can be embedded into $G(\Gamma)$.

The following is standard.

Proposition

Λ, Γ: finite graphs
If $\Lambda \leq \Gamma$, then $G(\Lambda) \hookrightarrow G(\Gamma)$.

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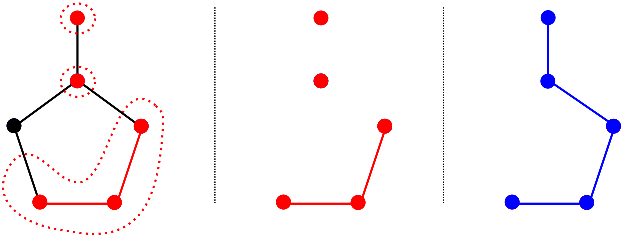
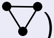
A subgraph Λ of a graph Γ is said to be full if $E(\Lambda)$ contains every $e \in E(\Gamma)$ whose end points both lie in $V(\Lambda)$.
We denote by $\Lambda \leq \Gamma$ if Λ is a full subgraph of Γ.



In general, the converse implication "$G(\Lambda) \hookrightarrow G(\Gamma)$" $\Rightarrow$ "$\Lambda \leq \Gamma$" is false.

Example

$G($$) \cong F_3 \hookrightarrow F_2 \cong G($$)$.

Proposition (cf. Charney-Vogtmann, 2009)

K_n^c: the edgeless graph on n vertices
Γ: a finite graph
Then $(\mathbb{Z}^n \cong) G(K_n^c) \hookrightarrow G(\Gamma)$ if and only if $K_n^c \leq \Gamma$.

In the case where $\Gamma = K_m^c$, the above theorem just says "$\mathbb{Z}^n \hookrightarrow \mathbb{Z}^m$ if and only if $n \leq m$".

Question

Which finite graph Λ satisfies the following property:
for any finite graph Γ, "$G(\Lambda) \hookrightarrow G(\Gamma)$" $\Rightarrow$ "$\Lambda \leq \Gamma$"?

The following gives a complete answer to the above question.

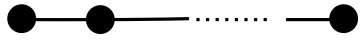
Theorem A (K.)

Let Λ be a finite graph.

(1) If Λ is a linear forest, then Λ has the above property, i.e., $\forall \Gamma$, if $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \Gamma$.

(2) If Λ is not a linear forest, then Λ does not have the above property, i.e., $\exists \Gamma$ such that $G(\Lambda) \hookrightarrow G(\Gamma)$, though $\Lambda \not\leq \Gamma$.

A finite graph Λ is said to be a linear forest if each connected component of Λ is a path graph.


P_n

P_n: the path graph consisting of n vertices

Theorem A(1)

Suppose that Λ is a linear forest.
Then $\forall\Gamma$, $G(\Lambda) \hookrightarrow G(\Gamma)$ implies $\Lambda \leq \Gamma$.

Application of Thm A(1) to concrete embedding problems

- $\neg(\mathbb{Z}^2 * \mathbb{Z} \hookrightarrow F_2 \times F_2 \times \cdots \times F_2)$.

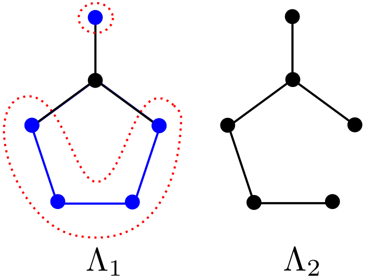
Note: $G(P_3) \cong \mathbb{Z}^2 * \mathbb{Z}$, $G(P_2 \sqcup \cdots \sqcup P_2) \cong F_2 \times \cdots \times F_2$.

Proof) Suppose to the contrary that $\mathbb{Z}^2 * \mathbb{Z} \hookrightarrow F_2 \times F_2 \times \cdots \times F_2$.
Then since P_3 is a linear forest, Theorem A(1) implies
$P_3 \leq P_2 \sqcup P_2 \sqcup \cdots \sqcup P_2$, a contradiction. Q.E.D.

Appl of Thm A(1) (cont'd).

• $\neg(G(\Lambda_1) \hookrightarrow G(\Lambda_2))$.

Proof) Suppose to the contrary that $G(\Lambda_1) \hookrightarrow G(\Lambda_2)$.
Then since $P_1 \sqcup P_4 \leq \Lambda_1$, we have $G(P_1 \sqcup P_4) \hookrightarrow G(\Lambda_1)$.
Hence, $G(P_1 \sqcup P_4) \hookrightarrow G(\Lambda_2)$.
This together with Theorem A(1) implies $P_1 \sqcup P_4 \leq \Lambda_2$, which is impossible. Q.E.D.



Theorem A(1) is sometimes valid to find that the RAAG, on a graph which is not a linear forest, cannot embed into another RAAG.

Moreover, we obtain the following as a consequence of Theorem A(1).

Theorem

Λ: a linear forest

If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, then $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

This is a partial converse of the following embedding theorem.

Theorem (Koberda, 2012)

Λ: a finite graph

If $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$, then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$

Proof of Theorem A(1)

Theorem A(1)

Λ: a linear forest

Γ: a finite graph

If $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \Gamma$.

Sketch of proof.

Step 1. Prove $\Lambda \leq \overline{\Gamma^e}$, where $\overline{\Gamma^e}$ is a graph such that

- $V(\overline{\Gamma^e}) = \{g^{-1}ug \in G(\Gamma) \mid u \in V(\Gamma),\ g \in G(\Gamma)\}$.
- u^g and v^h span an edge $\Leftrightarrow$ u^g and v^h are not commutative.

Theorem (Casals-Ruiz, 2015)

For a forest Λ and a finite graph Γ, if $G(\Lambda) \hookrightarrow G(\Gamma)$, then $\Lambda \leq \overline{\Gamma^e}$.

Step 2. Prove that $\Lambda \leq \overline{\Gamma^e}$ implies $\Lambda \leq \Gamma$.

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Use the "finiteness" of $\overline{\Gamma^e}$.

Theorem (Kim-Koberda, 2013)

If $\Lambda \leq \overline{\Gamma^e}$, then there exists a sequence of consecutive "co-doubles"

$$\Gamma = \Gamma_0 \leq \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_n \leq \overline{\Gamma^e}$$

such that $\Gamma_i = \overline{D}(\Gamma_{i-1})$ and $\Lambda \leq \Gamma_n$.

Here, for a finite graph Δ,

$$\overline{D}(\Delta) := (D(\Delta^c))^c.$$

The operation c: "taking the complement graph"
The operation D: "taking a double graph"

Step 2. Prove that $\Lambda \leq \overline{\Gamma^e}$ implies $\Lambda \leq \Gamma$ (cont'd).

Use the "finiteness" of $\overline{\Gamma^e}$.

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If $\Lambda \leq \overline{\Gamma^e}$, then there exists a sequence of consecutive "co-doubles"

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such that $\Gamma_i = \overline{D}(\Gamma_{i-1})$ and $\Lambda \leq \Gamma_n$.

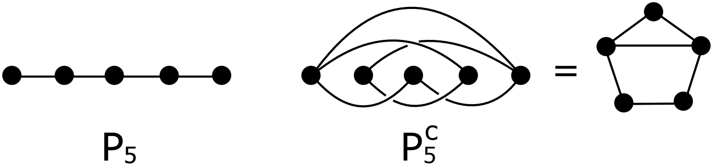
Proposition (K.)

Λ: a linear forest
Δ: a finite graph
If $\Lambda \leq \overline{D}(\Delta)$, then $\Lambda \leq \Delta$.

This completes the proof.

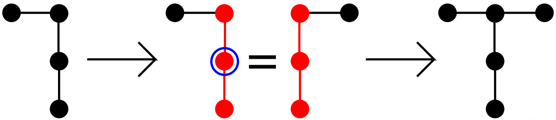
For a graph Γ, the complement graph Γ^c is the graph consisting of

- $V(\Gamma^c) = V(\Gamma)$ and
- $E(\Gamma^c) = \{\{u, v\} \mid u, v \in V(\Gamma), \{u, v\} \notin E(\Gamma)\}$.



$\mathrm{St}(v, \Gamma)$: the full subgraph induced by v and the vertices adjacent to v.

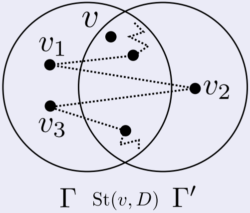
$D_v(\Gamma)$: the double of Γ along the full subgraph $\mathrm{St}(v, \Gamma)$, namely, $D_v(\Gamma)$ is obtained by taking two copies of Γ and gluing them along copies of $\mathrm{St}(v, \Gamma)$.

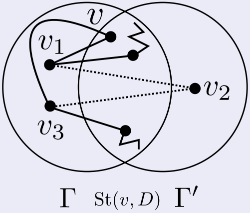


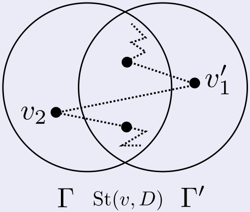
Proposition

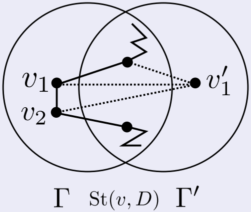
Λ: the complement graph of a linear forest, Γ: a finite graph
If $\Lambda \leq D_v(\Gamma)$, then $\Lambda \leq \Gamma$.

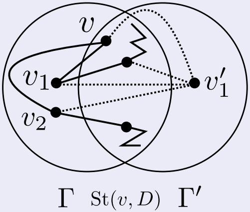
Example











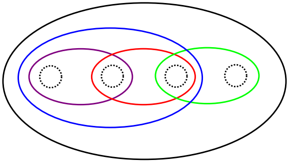
RAAGs in mapping class groups—future work—

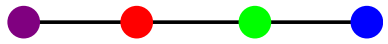
$\Sigma_{g,n}$: the orientable compact surface of genus g with n punctures
We assume $\chi(\Sigma_{g,n}) < 0$.
The mapping class group of $\Sigma_{g,n}$ is defined as follows.

$$\mathcal{M}(\Sigma_{g,n}) := \pi_0(Homeo^+(\Sigma_{g,n}))$$

The complement graph of the curve graph $\mathcal{C}^c(\Sigma_{g,n})$ is a graph such that

- $V(\mathcal{C}^c(\Sigma_{g,n})) = \{$isotopy classes of esls on $\Sigma_{g,n}\}$
- esls α, β span an edge iff α, β CANNOT be realized disjointly.





Theorem (Koberda, 2012)

Λ: a finite graph
If $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$, $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$.

Theorem (K.)

Λ: a linear forest
If $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$, then $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

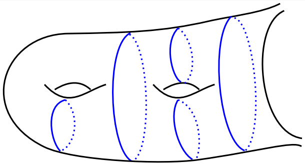
Theorem (Koberda + K.)

Λ: a linear forest
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

We can regard the above theorem as a generalization of the following classical result.

Theorem (Birman-Lubotzky-McCarthy, 1983)

$\mathbb{Z}^n \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if n does not exceed the number of simple closed curves needed in the pants-decomposition of $\Sigma_{g,n}$ $(= 3g + n - 3)$.



Theorem (Koberda + K.)

Λ: a linear forest
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

Theorem (BLM in our terminology)

Λ: the disjoint union of finitely many copies of P_1
Then $G(\Lambda) \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if $\Lambda \leq \mathcal{C}^c(\Sigma_{g,n})$.

Bering IV, Conant and Gaster proved that
$P_2 \sqcup P_2 \sqcup \cdots \sqcup P_2 \leq \mathcal{C}^c(\Sigma_{g,n})$ if and only if the number of the copies of P_2 is at most $g + \lfloor \frac{g+n}{2} \rfloor - 1$ in this September...

Proposition

$F_2 \times F_2 \times \cdots \times F_2 \hookrightarrow \mathcal{M}(\Sigma_{g,n})$ if and only if the number of the direct factors F_2 is at most $g + \lfloor \frac{g+n}{2} \rfloor - 1$.

Question (Kim-Koberda, 2014)

Given a right-angled Artin group, what is the simplest surface for which there is an embedding of the right-angled Artin group into the mapping class group?

e.g. for $F_2 \times F_2 \times F_2$, the simplest surface(s) are $\Sigma_{2,2}$, $\Sigma_{3,0}$...

References

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This talk is based on:

• T. Katayama, 'Right-angled Artin groups and full subgraphs of graphs', preprint, available at arXiv:1612.01732.

Thank you very much for your attention!