ハンドル体結び目の(同辺)結び目解消数とAlexanderバイカンドルのG族彩色

筑波大学数理物質科学研究科 D1

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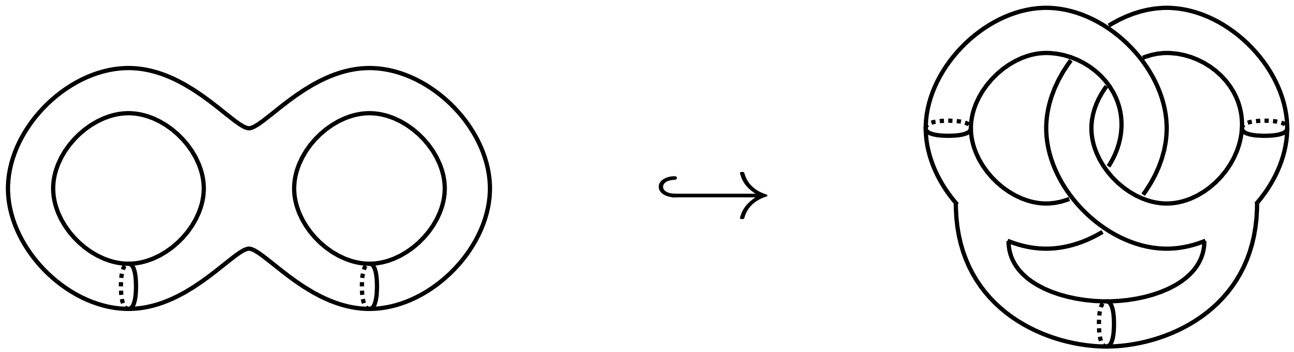
Def

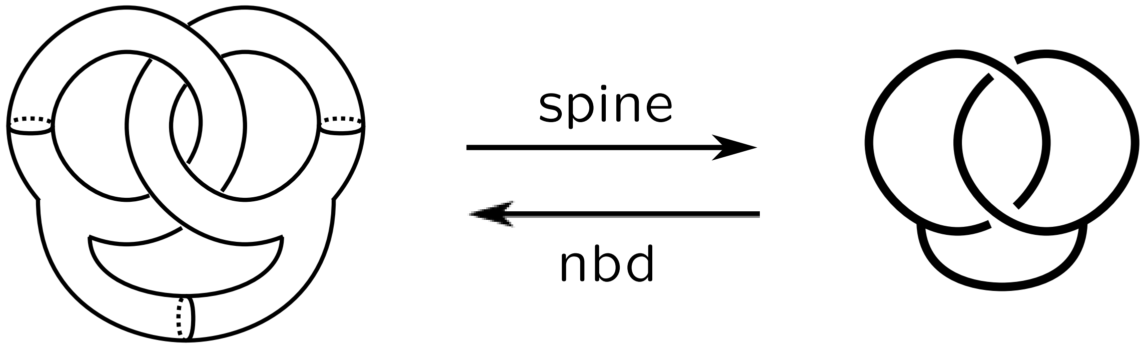
- **handlebody-knot** : handlebody $\hookrightarrow S^3$



- H_1, H_2 : handlebody-knot

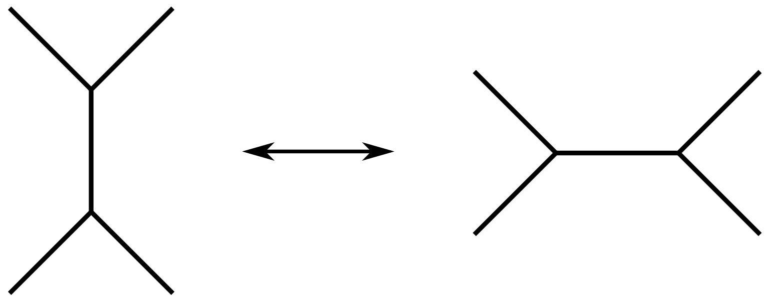
$H_1 \cong H_2 \overset{\text{def}}{\Longleftrightarrow} \exists f : S^3 \to S^3$ orientator preserving homeo.

s.t. $f(H_1) = H_2$

- 

handlebody-knot spatial trivalent graph

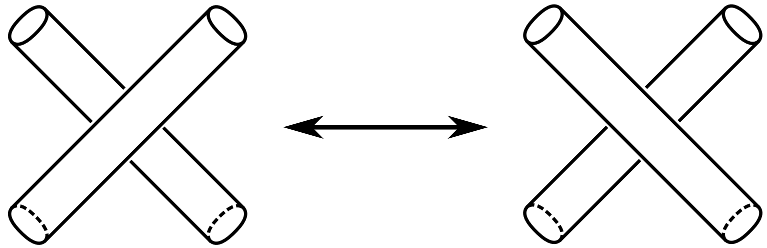
- spatial trivalent graphの**IH-move**



<u>Rem</u>

$\{\text{hdbdy-knot}\}/\cong \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{sp. tri. graph}\}/\cong, \text{IH-move}$

- handlebody-knotの**crossing change**



- handlebody-knotの**self-edge crossing change**

$\overset{\text{def}}{\Longleftrightarrow}$ hdbdy-knotを表すsp. tri. graphにおける同じ辺同士のcrossing change.

Lem

任意の同種数hdbdy-knotは有限回の (self-edge) crossing changeで移り合う.

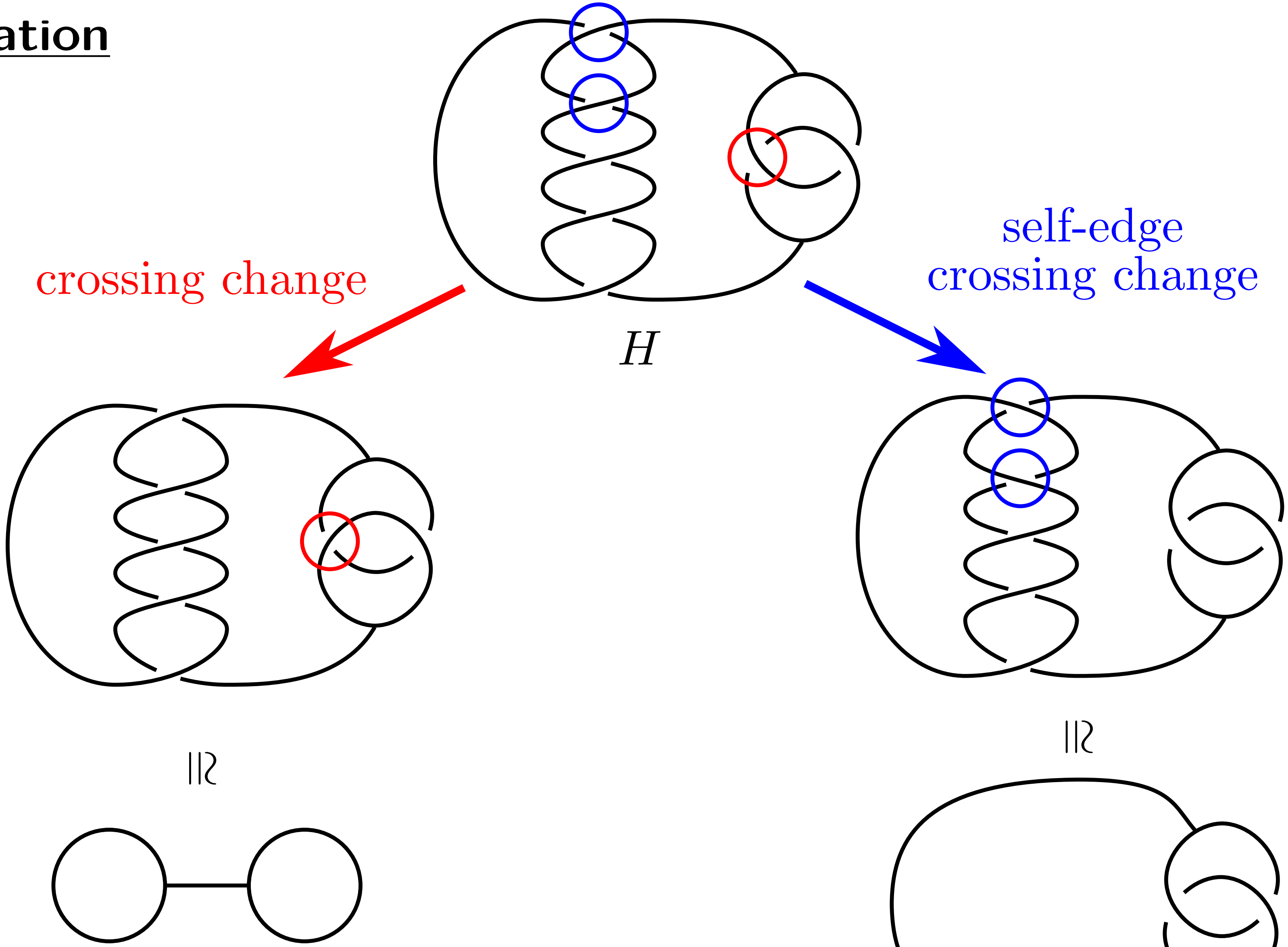
Def

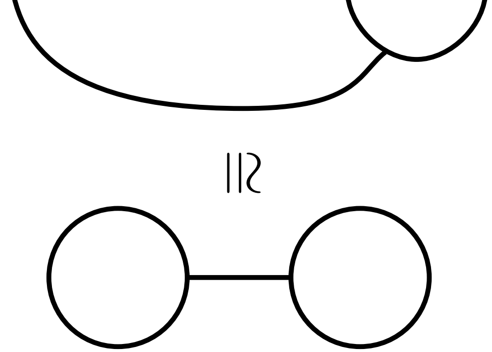
H_1, H_2 : 種数gのhandlebody-knot
O_g : 種数gのtrivial handlebody-knot

- $d(H_1, H_2) := \min\{n \mid H_1 \leftrightarrow (n$回のcrossing change$) \leftrightarrow H_2\}$
  : H_1とH_2の**Gordian distance**
  $u(H_1) := d(H_1, O_g)$: H_1の**unknotting number**

- $\overline{d}(H_1, H_2) := \min\{n \mid H_1 \leftrightarrow (n$回のself-edge crossing change$) \leftrightarrow H_2\}$
  : H_1とH_2の**self-edge Gordian distance**
  $\overline{u}(H_1) := \overline{d}(H_1, O_g)$: H_1の**self-edge unknotting number**

Motivation





$\therefore\ \boldsymbol{u(H) = 1}$

$\quad\boldsymbol{\overline{u}(H) = 2}$?

Main result

(1) handlebody-knotのGordian distance及びunknotting numberに関する評価式を得た.

(2) handlebody-knotのself-edge Gordian distance及びself-edge unknotting numberに関して，(1)よりも強い評価式を得た.

(3) あるhandlebody-knotに対し，Gordian distance及びunknotting numberを決定した.

Def

$(X, *)$: quandle

$\overset{\text{def}}{\Longleftrightarrow}$

• $x * x = x \ (\forall x \in X)$

• $*\, x : X \to X; y \mapsto y * x$: bijection $(\forall x \in X)$

• $(x * y) * z = (x * z) * (y * z) \ (\forall x, y, z \in X)$

Def

$(X, \underline{*}, \overline{*})$: **biquandle**

$\overset{\text{def}}{\Longleftrightarrow}$

• $x \,\underline{*}\, x = x \,\overline{*}\, x \ (\forall x \in X)$

• $\underline{*}\, x : X \to X; y \mapsto y \,\underline{*}\, x$: bijection $(\forall x \in X)$

$\overline{*}\, x : X \to X; y \mapsto y \,\overline{*}\, x$: bijection $(\forall x \in X)$

$S : X \times X \to X \times X; (x, y) \mapsto (y \,\overline{*}\, x, x \,\underline{*}\, y)$: bijection

• $(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) = (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z)$

$(x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z)$

$(x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z) \ (\forall x, y, z \in X)$

Ex

$\mathbb{Z}_m[t^{\pm 1}, s^{\pm 1}]$: **Alexander biquandle**

$$(x \underline{*} y := tx + (s-t)y,\ x \overline{*} y := sx)$$

特に，p : prime, $s \in \mathbb{Z}_p[t^{\pm 1}]$, $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$: irr. poly. のとき，
$\mathbb{Z}_p[t^{\pm 1}, s^{\pm 1}]/(f(t))$: Alexander biquandle, (field)

Def

$(X, \underline{*}, \overline{*})$: biquandle

• $x \underline{*}^{[n]} y$ $(n \in \mathbb{Z})$ を次で定義する($x \overline{*}^{[n]} y$ についても同様に定義).

$x \underline{*}^{[0]} y := x$
$x \underline{*}^{[1]} y := x \underline{*} y$
$x \underline{*}^{[2]} y := (x \underline{*} y) \underline{*} (y \underline{*} y)$
$x \underline{*}^{[i+j]} y := (x \underline{*}^{[i]} y) \underline{*}^{[j]} (y \underline{*}^{[i]} y)$

• $\mathrm{type}(X) := \min\{n > 0 \mid x \underline{*}^{[n]} y = x = x \overline{*}^{[n]} y\ (\forall x, y \in X)\}$

Def

G : group とする．このとき，

$(X, \{\underline{*}^g\}_{g\in G}, \{\overline{*}^g\}_{g\in G})$: **G-family of biquandles**

$\overset{\text{def}}{\Longleftrightarrow}$

(i) $x \,\underline{*}^g\, x = x \,\overline{*}^g\, x \ (\forall g \in G, \forall x \in X)$

(ii) $\underline{*}^g x : X \to X; y \mapsto y \,\underline{*}^g\, x$: bijection $(\forall g \in G, \forall x \in X)$

$\overline{*}^g x : X \to X; y \mapsto y \,\overline{*}^g\, x$: bijection $(\forall g \in G, \forall x \in X)$

$S_{g,h} : X \to X; (x, y) \mapsto (y \,\overline{*}^g\, x, x \,\underline{*}^h\, y)$: bijection $(\forall g, h \in G)$

(iii) $(x \,\underline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\underline{*}^h\, z) \,\underline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z)$

$(x \,\overline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\underline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z)$

$(x \,\overline{*}^g\, y) \,\overline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\overline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z)$

$(\forall g, h \in G, \forall x, y, z \in X)$

(iv) $x \,\underline{*}^{gh}\, y = (x \,\underline{*}^g\, y) \,\underline{*}^h\, (y \,\underline{*}^g\, y)$

$x \,\overline{*}^{gh}\, y = (x \,\overline{*}^g\, y) \,\overline{*}^h\, (y \,\overline{*}^g\, y) \quad (\forall g, h \in G, \forall x, y \in X)$

Ex

$(X, \underline{*}, \overline{*})$: biquandle, $k := \mathrm{type}(X)$

このとき，

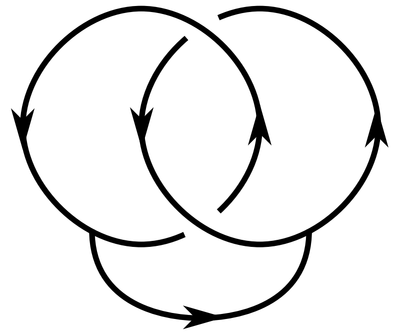
$(X, \{\underline{*}^{[n]}\}_{[n]\in\mathbb{Z}_k}, \{\overline{*}^{[n]}\}_{[n]\in\mathbb{Z}_k})$: **$\mathbb{Z}_k$-family of biquandles**

Def

H : handlebody-knot

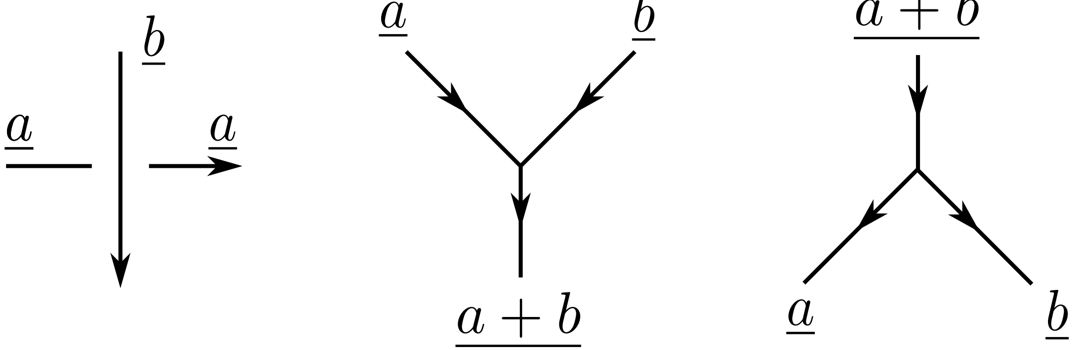
D : Hの（Y-oriented） diagram

- $\phi : \mathcal{A}(D) = \{D\text{の arc}\} \to \mathbb{Z}_k$: Dの$\mathbb{Z}_k$-**flow**



D

$\overset{\text{def}}{\Longleftrightarrow}$

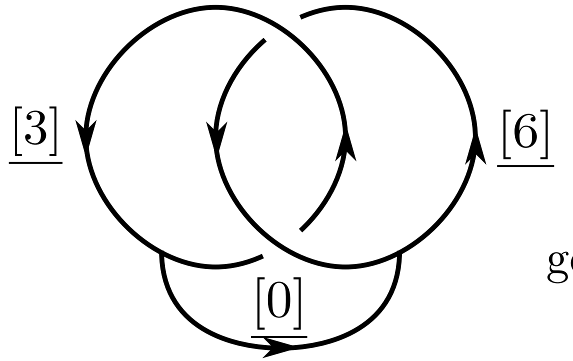


$a, b \in \mathbb{Z}_k$

- $\mathbb{Z}_k$-flow ϕ の与えられたdiagram D を (D, ϕ) で表す.

- ϕ : Dの$\mathbb{Z}_k$-flowに対し,

$\mathbf{gcd}(\phi) := \mathbf{gcd}\{\phi(a), k \mid a \in \mathcal{A}(D)\}$



$\gcd(\phi) = 3$

(D, ϕ) ϕ : D の $\mathbb{Z}_9$-flow

Def

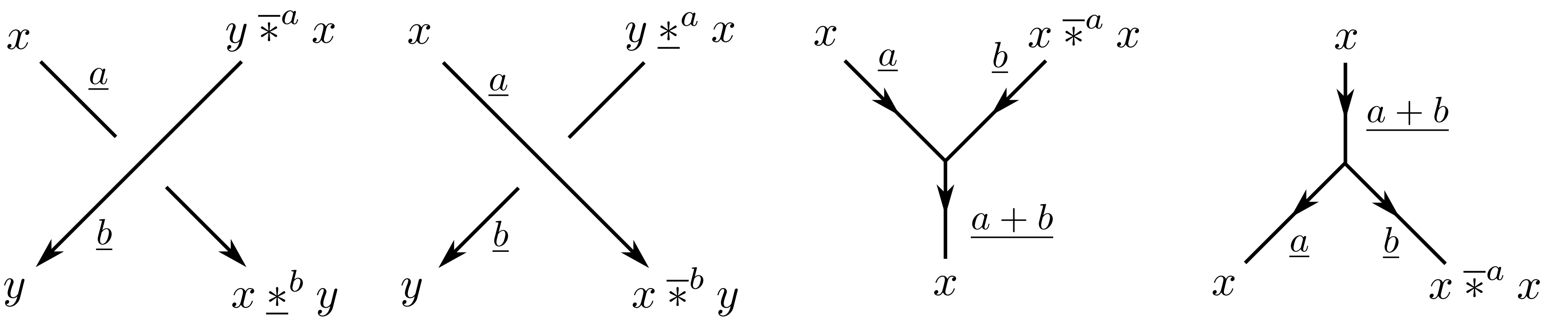
H : handlebody-knot

(D, ϕ) : H の $\mathbb{Z}_k$-flowed diagram

X : $\mathbb{Z}_k$-family of biquandles

• $C : \mathcal{SA}(D, \phi) = \{(D, \phi)$ の semi-arc$\} \to X$: (D, ϕ) の X-**coloring**

$\overset{\text{def}}{\Longleftrightarrow}$



$a, b \in \mathbb{Z}_k$
$x, y \in X$

• $\mathrm{Col}_X(D, \phi) := \{(D, \phi)$ の X-coloring$\}$

特に，X : field のとき，$\mathrm{Col}_X(D, \phi)$: vector space over X

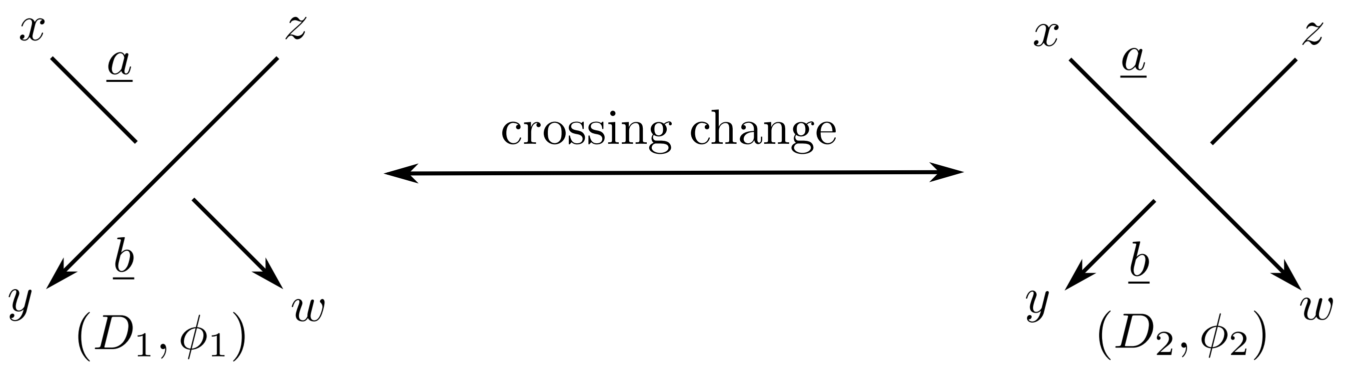
Main theorem[M]

H_1, H_2 : 種数 g の handlebody-knot
(D_1, ϕ_1) : H_1 の $\mathbb{Z}_k$-flowed diagram
D_2 : H_2 の diagram
p : prime, $s \in \mathbb{Z}_p[t^{\pm 1}]$, $f(t) \in \mathbb{Z}_p[t^{\pm 1}]$: irr. poly.
$X = \mathbb{Z}_p[t^{\pm 1}, s^{\pm 1}]/(f(t))$: $\mathbb{Z}_k$-family of Alexander biquandles, (field)
このとき,

- $$\min_{\substack{\phi_2 : D_2 \text{ の } \mathbb{Z}_k\text{-flow} \\ \gcd(\phi_1) = \gcd(\phi_2)}} \frac{|\dim \mathrm{Col}_X(D_1, \phi_1) - \dim \mathrm{Col}_X(D_2, \phi_2)|}{2} \leq d(H_1, H_2)$$
$$\min_{\substack{\phi_2 : D_2 \text{ の } \mathbb{Z}_k\text{-flow} \\ \gcd(\phi_1) = \gcd(\phi_2)}} |\dim \mathrm{Col}_X(D_1, \phi_1) - \dim \mathrm{Col}_X(D_2, \phi_2)| \leq d(H_1, H_2) \quad (s = 1, t)$$
$$\min_{\substack{\phi_2 : D_2 \text{ の } \mathbb{Z}_k\text{-flow} \\ \gcd(\phi_1) = \gcd(\phi_2)}} |\dim \mathrm{Col}_X(D_1, \phi_1) - \dim \mathrm{Col}_X(D_2, \phi_2)| \leq \overline{d}(H_1, H_2)$$

- $$\frac{\dim \mathrm{Col}_X(D_1, \phi_1) - 1}{2} \leq u(H_1)$$
$$\dim \mathrm{Col}_X(D_1, \phi_1) - 1 \leq u(H_1) \quad (s = 1, t)$$
$$\dim \mathrm{Col}_X(D_1, \phi_1) - 1 \leq \overline{u}(H_1)$$

Proof



col. rel. $\begin{cases} ① \ w = t^b x + (s^b - t^b) y \\ ② \ z = s^a y \end{cases}$

col. rel. $\begin{cases} ①' \ w = s^b x \\ ②' \ z = t^a y + (s^a - t^a) x \end{cases}$

$$\mathbf{Col}_{\boldsymbol{X}}(\boldsymbol{D_1}, \boldsymbol{\phi_1}) = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \\ \vdots \end{pmatrix} \in \boldsymbol{X}^n \;\middle|\; \begin{pmatrix} ① & \text{———} \\ ② & \text{———} \\ ③ & \text{———} \\ & \vdots \\ ⓝ & \text{———} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \right\}$$

$\Updownarrow$ rank の差は高々 2

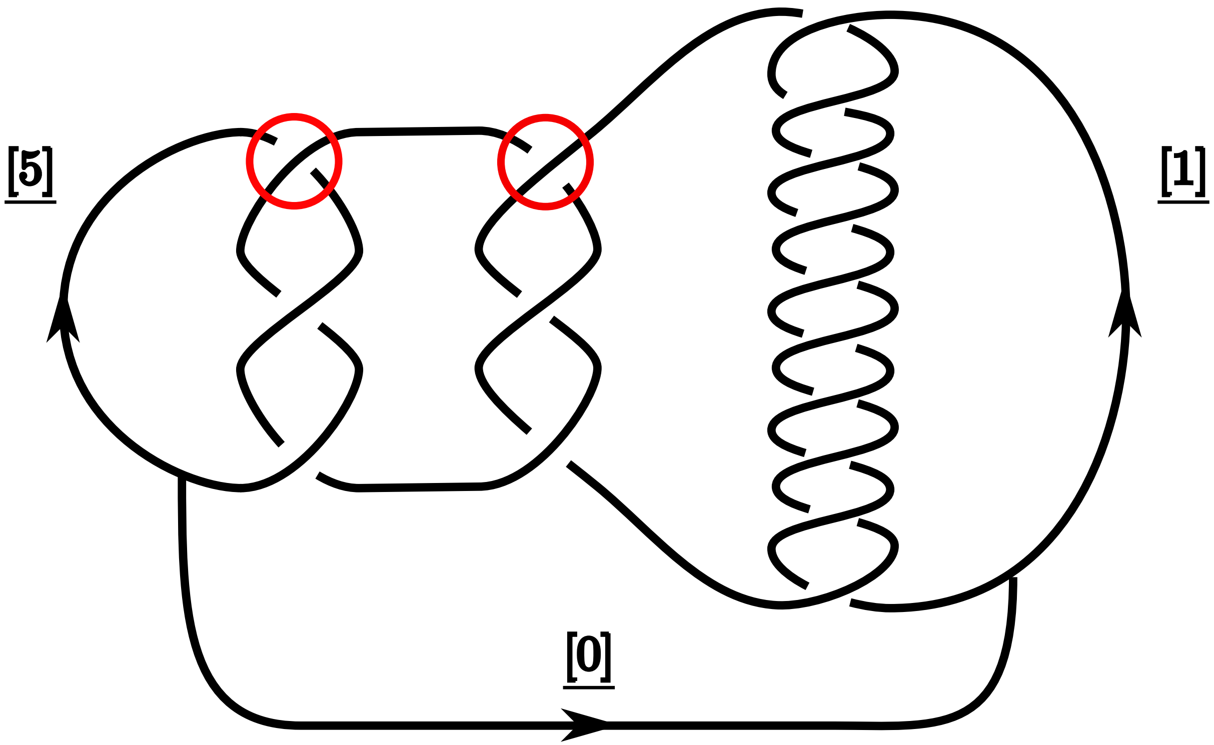
$$\mathbf{Col}_{\boldsymbol{X}}(\boldsymbol{D_2}, \boldsymbol{\phi_2}) = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \\ \vdots \end{pmatrix} \in \boldsymbol{X}^n \;\middle|\; \begin{pmatrix} ①' & \text{———} \\ ②' & \text{———} \\ ③ & \text{———} \\ & \vdots \\ ⓝ & \text{———} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \\ \boldsymbol{w} \\ \vdots \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix} \right\}$$

Ex

$s := 1$, $f(t) := 1 + 2t + t^2 + 2t^3 + t^4 \in \mathbb{Z}_3[t^{\pm 1}]$: irr. poly.

$X = \mathbb{Z}_3[t^{\pm 1}, s^{\pm 1}]/(f(t))$: $\mathbb{Z}_{10}$-family of Alexander biquandles, (field)

H :



(D, ϕ) $\phi : D$ の $\mathbb{Z}_{10}$-flow

$\dim \mathrm{Col}_X(D, \phi) = 3$

$\therefore \dim \mathrm{Col}_X(D, \phi) - 1 = 2 \leq u(H)$ ($\because$ Main theorem)

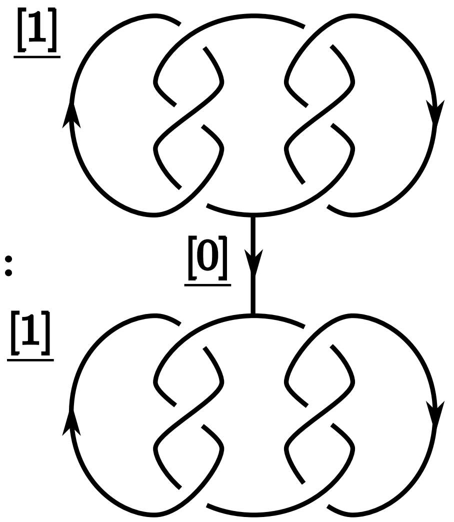
$\therefore u(H) = 2$

Ex

$s := 1$, $f(t) := 1 + t + t^2 \in \mathbb{Z}_2[t^{\pm 1}]$: irr. poly.

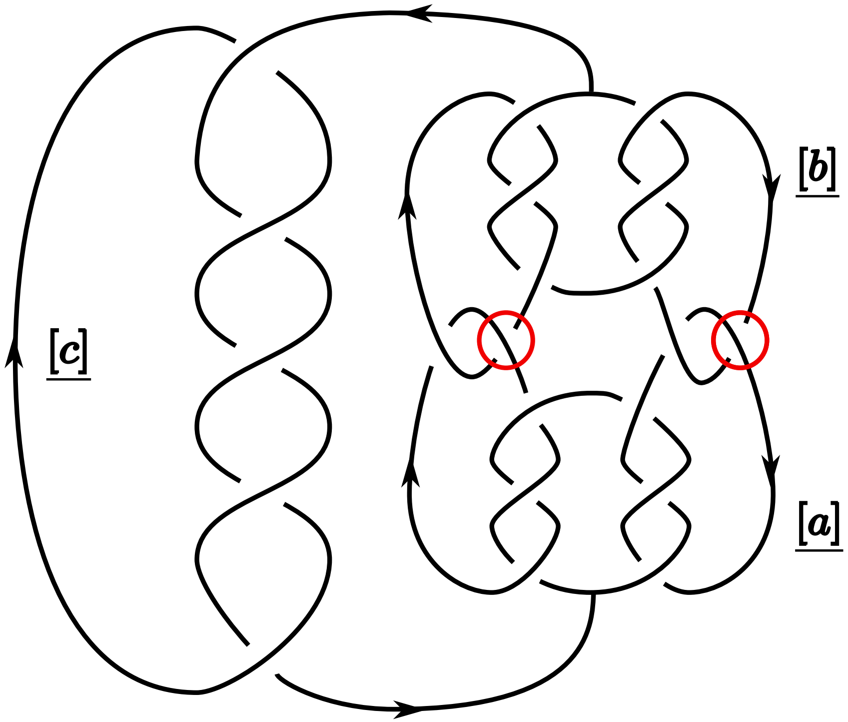
$X = \mathbb{Z}_2[t^{\pm 1}, s^{\pm 1}]/(f(t))$: $\mathbb{Z}_3$-family of Alexander biquandles, (field)

H_1 :



(D_1, ϕ_1) ϕ_1 : D_1 の $\mathbb{Z}_3$-flow

H_2 :



(D_2, ϕ_2) ϕ_2 : D_2 の $\mathbb{Z}_3$-flow

$\dim \mathrm{Col}_X(D_1, \phi_1) = 5$

一方，$\forall\ \phi_2$: D_2 の $\mathbb{Z}_3$-flow に対し，$\dim \mathrm{Col}_X(D_2, \phi_2) \leq 3$

$\therefore 2 \leq \dim \mathrm{Col}_X(D_1, \phi_1) - \dim \mathrm{Col}_X(D_2, \phi_2)$

$\therefore 2 \leq d(H_1, H_2)$ ($\because$ Main theorem)

$\therefore d(H_1, H_2) = 2$