The virtual unknotting numbers of a class of virtual torus knots

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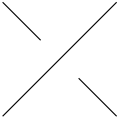
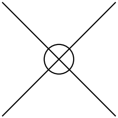
§1. Virtual knot

Definition (virtual knot diagram)

A virtual knot diagram is a diagram on $\mathbb{R}^2$ that has classical crossings and virtual crossings.



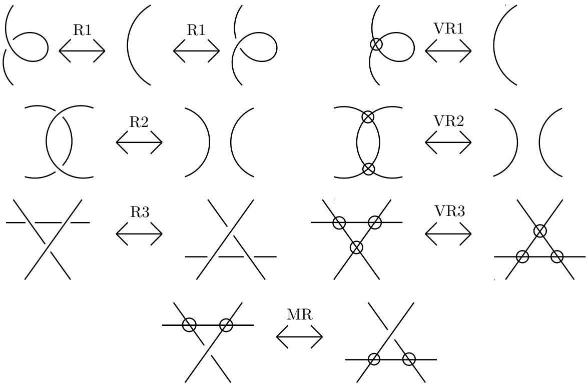
classical crossing



virtual crossing

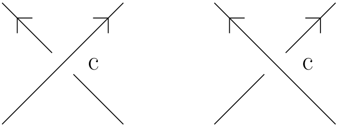
Definition

Virtual knot and link diagrams that can be connected by a finite sequence of generalized Reidemeister moves are said to be equivalent or virtually isotopic.



Definition

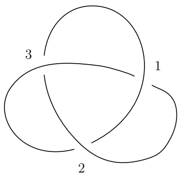
At a crossing point c of an oriented regular diagram, as shown below, we have two possible configurations. The crossing point on the left side is said to be positive, while that on the right side is said to be negative.

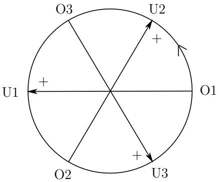


$sign(c) = +1$ $sign(c) = -1$

Definition (Gauss diagram)

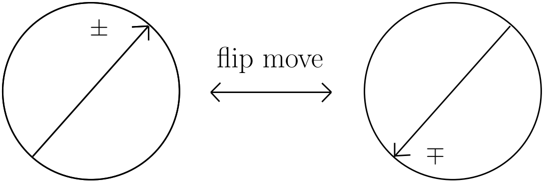
A knot diagram D gives rise to a Gauss diagram G_D that is a circle parameterizing the knot with each pair of preimages of double points of D connected by an oriented signed chord. The chords are oriented from the preimage point on the over-passing branch to the preimage point on the under-passing branch. The sign of a chord is the sign of the corresponding double point.



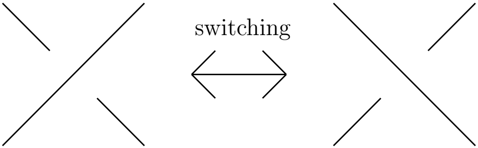


Definition

A (generic) homotopy between two Gauss diagrams is a sequence of isotopies and "flip moves" that change simultaneously the direction and the sign of an arrow.



A flip move on Gauss diagram is a switching on virtual knot.



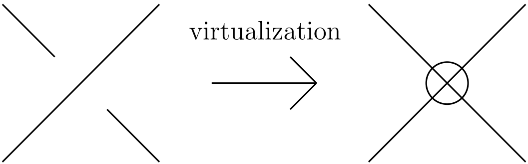
Definition

A virtual homotopy between two virtual knot diagrams is a sequence of virtually isotopies and switching.

Remark

In particular, not every virtual knot is homotopic to the unknot.

An operation which changes a classical crossing into a virtual crossing is called a virtualization. Any virtual knot can be modified into the unknot by switchings and virtualizations.



Definition (Kaur, Kamada, Kawauchi, Prabhakar, 2016)

Let K be a virtual knot. For each unknot sequence s, let c_s be the number of switchings and v_s be the number of virtualizations. Then the generalized unknotting number of K is defined as $U(K) = \min_s\{(v_s, c_s)\}$, where the minimality is taken with respect to lexiciographic order. For example, $\min\{(2,0),(0,1)\} = (0,1)$.

Definition

Suppose K is a virtual knot homotopic to the unknot. Then $U(K)$ has the form

$$U(K) = (0, c).$$

The second entry is called the virtual unknotting number of K and denoted by $\mathrm{vu}(K)$.

Definition (Torus knot)

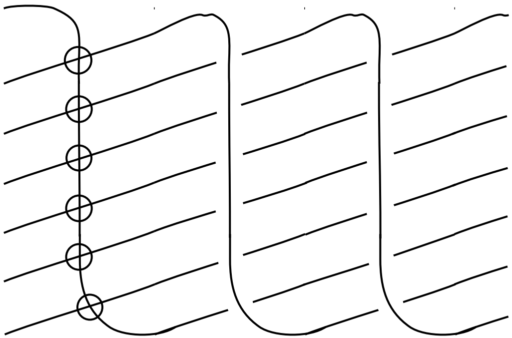
A (p, q) torus knot ($p, q \in \mathbb{N}$, p and q are coprime.) is a curve on the standard torus in S^3 that rotates p times along the longitude and q times along the meridian.

Definition (Virtual torus knot)

Let D be the standard diagram of the (p, q) torus knot. Let $a_1, a_2, \dots, a_q$ be the overstrands of D labelled in the canonical order. The virtual torus knot $VT^k_{p,q}$, $k \in \{1, \dots, q\}$, is the virtual knot obtained from D by changing the classical crossings on $a_1, a_2, \dots, a_k$ to virtual crossings.

Example

The diagram of $VT^{1}_{7,3}$ is the closure of the following diagram.



§2. Main result

Theorem (Y.)

The generalized unknotting number of $VT^1_{p,q}$ is

$$U(VT^1_{p,q}) = (0, \frac{(p-1)(q-1)}{2}).$$

In other words, $VT^1_{p,q}$ is homotopic to the unknot and $\mathrm{vu}(VT^1_{p,q}) = (p-1)(q-1)/2$.

Theorem (Kronheimer-Mrowka, '93)

For a torus knot $T_{p,q}$,

$$\mathrm{u}(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

§3. P-invariant and lower bound

Definition (Invariant $P(K)$ of A. Henrich)

Let D be a Gauss diagram realizing a virtual knot K. The end points of chord c in D separate the circle in D into two arcs. Choose one of the arcs and perform the flip moves on all the other chords so that they point into the chosen arc. Let $i(c)$ be the sum of the sign of the chords pointing into the chosen arc after the flip move. We put $\text{sign}(c) = \pm 1$ to be the sign of c.
Put

$$P(D) = \sum_{c \text{ such that } i(c) \neq 0} \text{sign}(c) t^{|i(c)|}$$

to be the polynomial in variable t with integer coefficients. One can show that $P(D)$ does not depend on the choice of diagram D realizing K. Hence we have a polynomial invariant $P(K)$ of virtual knot K.

Lemma (The P invariant and the virtual unknotting numbers)

Let K be a virtual knot homotopic to the unknot. Let $P(K) = \sum_{j>0} b_j t^j$. Then

$$vu(K) \geq \frac{\sum_{j>0} |b_j|}{2}.$$

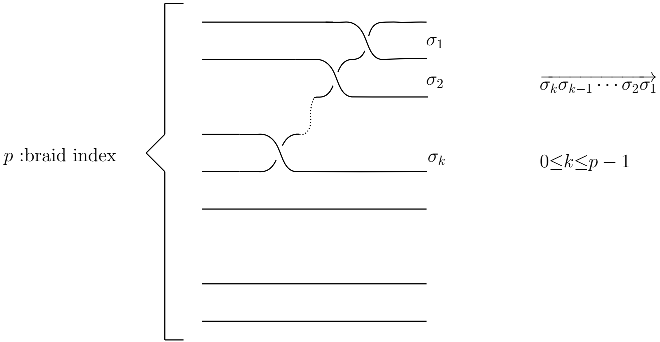
<u>Proof of lower bound of the main theorem</u>

Remark that $i(c) \neq 0$ for any crossing c of $VT^1_{p,q}$.
Hence $\sum_{j>0} |b_j| = \#$ classical crossing $= (p-1)(q-1)$.
Then vu$(VT^1_{p,q}) \geq \frac{(p-1)(q-1)}{2}$. □

§4. Upper bound

Definition (Braid B_k)

Let B_k denote the braid as follows.



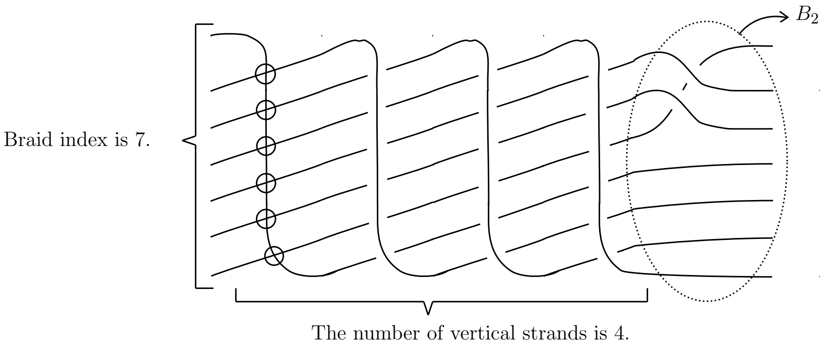
Let $VB^1_{i,j}$ be the virtual braid diagram obtained from the braid of (i,j)-torus link part changing all classical crossings on the left-most overstrand into virtual crossings. Note that i and j are not necessarily coprime.

Definition

We denote the closure of the product of $VB^1_{i,j}$ and B_k ($k \in \mathbb{Z}_{\geq 0}$ and $i > k$) by (i,j,k).

Example

(7,4,2) is shown below.



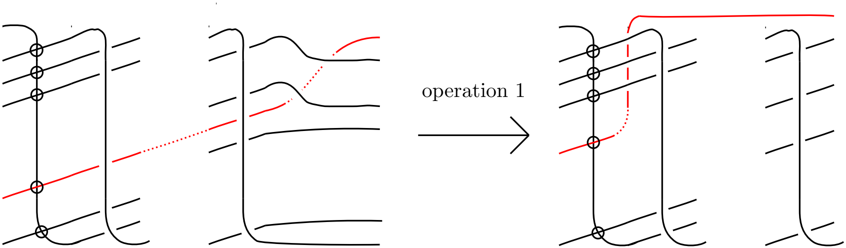
From now on we suppose that (i, j, k) is a virtual knot.

Proposition

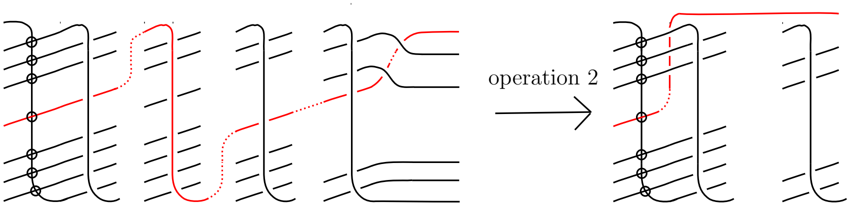
$$\mathrm{vu}((i,j,k)) \leq \frac{(i-1)(j-1)+k}{2}$$

$$(i,j,k) \underbrace{\to \cdots \to}_{\text{operations}} \text{ unknot}$$

$$\#\ \text{switching} = \frac{(i-1)(j-1)+k}{2}$$

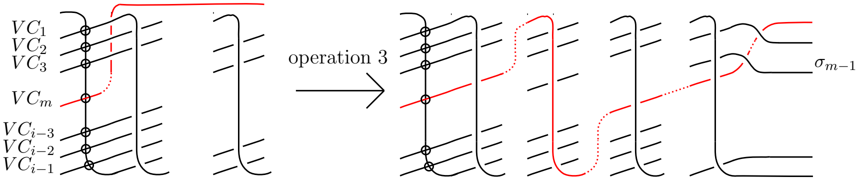


Operation 1
Choose the strand passing through the right-top, then move that strand as in the figure.

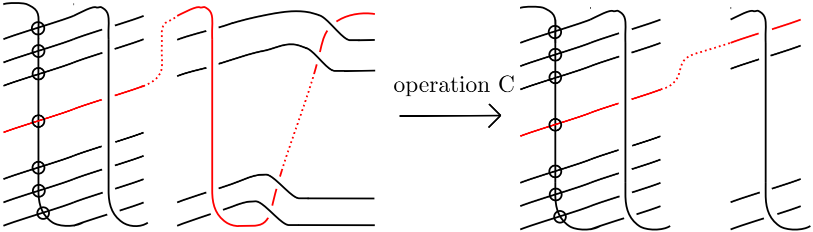


Operation 2

Choose the strand passing through the right-top, then move that strand as in the figure. (We need switchings.)



Operation 3
Apply VR1 after MR-moves, then isotope the classical crossings on the left of the virtual crossings to the right-most position (by conjugation of braid).



Operation C

Reduce overstrands by switchings.

Algorithm to obtain the unknot.

Operation A : operation 1 + operation 3
Operation B : operation 2 + operation 3

Suppose $i \geq j$.
If $k + j \leq i - 1$ then

$$(i, j, k) \overset{\text{operation A}}{\longrightarrow} (i - 1, j, j + k - 1)$$

If $i + 1 \leq k + j \leq 2i - 2, i \neq k + 1$ then

$$(i, j, k) \overset{\text{operation B}}{\longrightarrow} (i - 1, j - 1, j + k - i - 1)$$

If $i = k + 1$ then

$$(i, j, k) \overset{\text{operation C}}{\longrightarrow} (i, j - 1, 0)$$

We prove the upper bound by induction.
Let $m(i,j,k)$ be the number of operations A, B and C to obtain the unknot.
(i) Assume that the inequlaity holds for any virtual knot (i,j,k) with $m(i,j,k) \leq \ell$.
(ii) Check the inequality for a virtual knot (i',j',k') with $m(i',j',k') \leq \ell + 1$.

(A) Suppose that (i,j,k) is obtained from (i',j',k') by op. A.
In this case, $(i',j',k') = (i+1, j, k-j+1)$.
Then we have

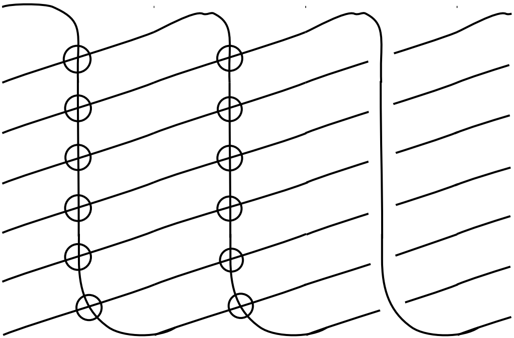
$$\frac{(i'-1)(j'-1)+k'}{2} = \frac{(i+1-1)(j-1)+k-j+1}{2} = \frac{(i-1)(j-1)+k}{2}. \qquad \square$$

Consider $VT^1_{14,5}$
(14, 5, 0)
(13, 5, 4) A, 0 times
(12, 5, 8) A, 0 times
(11, 4, 0) B, 11 times
(10, 4, 3) A, 0 times
(9, 4, 6) A, 0 times
(8, 3, 0) B, 8 times
(7, 3, 2) A, 0 times
(6, 3, 4) A, 0 times
(5, 2, 0) B, 5 times
(4, 2, 1) A, 0 times
(3, 2, 2) A, 0 times
(3, 1, 0) C, 2 times
The total number of switchings is 26.

§5. On other (p, q)-virtual torus knots

Example

The diagram of $VT^2_{7,3}$ is the closure of the following diagram.



Theorem (Y.)

Suppose p is odd. If $VT^2_{p,q}$ is homotopic to the unknot, the virtual unknotting number of $VT^2_{p,q}$ satisfies $\mathrm{vu}(VT^2_{p,q}) \geq \frac{(p-1)(q-2)}{2}$.

If p is even, we can obtain a lower bound by P -invariant for each example, though we could not find a general formula.

$VT^2_{p,q}$	lower bound	upper bound	result
(3,2)	0	0	vu$(VT^2_{3,2}) = 0$
(4,3)	1	1	vu$(VT^2_{4,3}) = 1$
(5,2)	0	0	vu$(VT^2_{5,2}) = 0$
(5,3)	2	2	vu$(VT^2_{5,3}) = 2$
(5,4)	4	4	vu$(VT^2_{5,4}) = 4$
(6,5)	7	7	vu$(VT^2_{6,5}) = 7$
(7,2)	0	0	vu$(VT^2_{7,2}) = 0$
(7,3)	3	3	vu$(VT^2_{7,3}) = 3$
(7,4)	6	6	vu$(VT^2_{7,4}) = 6$
(7,5)	9	9	vu$(VT^2_{7,5}) = 9$
(7,6)	12	12	vu$(VT^2_{7,6}) = 12$
(8,3)	3	3	vu$(VT^2_{8,3}) = 3$
(8,5)	9	10	$9 \leq$ vu$(VT^2_{8,5}) \leq 10$
(8,7)	17	17	vu$(VT^2_{8,7}) = 17$

$VT^3_{p,q}$	lower bound	upper bound	result
(4,3)	0	0	vu$(VT^3_{4,3}) = 0$
(5,3)	0	0	vu$(VT^3_{5,3}) = 0$
(5,4)	2	2	vu$(VT^3_{5,4}) = 2$
(6,5)	4	4	vu$(VT^3_{6,5}) = 4$
(7,3)	0	0	vu$(VT^3_{7,3}) = 0$
(7,4)	3	3	vu$(VT^3_{7,4}) = 3$
(7,5)	6	6	vu$(VT^3_{7,5}) = 6$
(7,6)	9	9	vu$(VT^3_{7,6}) = 9$
(8,3)	0	0	vu$(VT^3_{8,3}) = 0$
(8,5)	7	7	vu$(VT^3_{8,5}) = 7$
(8,7)	14	14	vu$(VT^3_{8,7}) = 14$