Generating the mapping class group of a surface by torsion

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Orientable surface

$\Sigma_{g,n}$: a closed orientable surface of genus g with arbitrarily chosen n points.

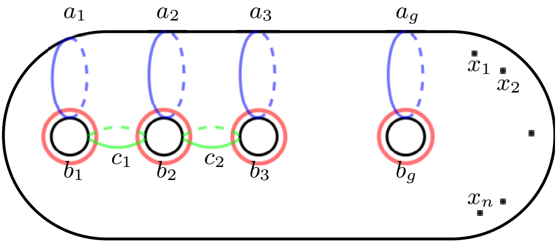
we call punctures $P = \{x_1, x_2, \cdots, x_n\}$

$\mathrm{Diff}^+(\Sigma_{g,n}) := \{f : \Sigma_{g,n} \to \Sigma_{g,n}$ |orientation preserving differomorphism, $f(P) = P\}$

$\mathrm{Diff}_0(\Sigma_{g,n}) := \{f \in \mathrm{Diff}^+(\Sigma_{g,n}) \mid f$ is isotopic to identity $\}$

$\mathrm{Mod}(\Sigma_{g,n}) := \mathrm{Diff}^+(\Sigma_{g,n})/\mathrm{Diff}_0(\Sigma_{g,n})$

: the mapping class group of $\Sigma_{g,n}$

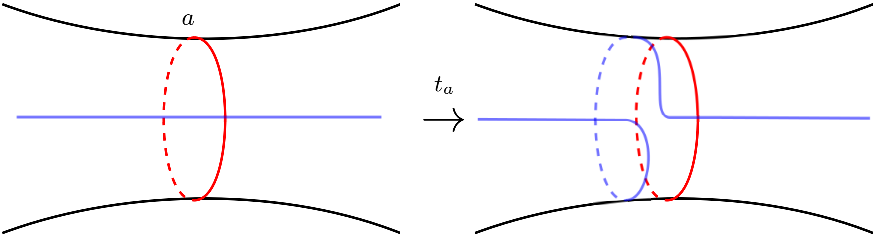


closed orientable surface

Dehn twist

a : simple closed curve on $\Sigma_{g,n}$.

$t_a :=$ the Dehn twist along a



The Dehn twist along a

relation for $\mathrm{Mod}(\Sigma_{g,n})$

Lemma 1.1

a : a simple closed curve on $\Sigma_{g,n}$
For $f \in \mathrm{Mod}(\Sigma_{g,n})$,

$$f t_a f^{-1} = t_{f(a)}.$$

an ordered set of $c_1, c_2, \ldots, c_n$ of simple closed curves on Σ_g forms *n-chain*
$\iff$ c_i and c_{i+1} intersect transversely at one point for $i = 1, 2, \ldots, n-1$ and c_i is disjoint from c_j if $\mid i - j \mid \geq 2$.
If n is odd, the boundary of regular neighborhood of n-chain has two components d_1 and d_2.

Lemma 1.2

$\{c_1, c_2, c_3, c_4, c_5\}$: chain on $\Sigma_{g,n}$
we have following relation.

$$(t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_{c_5})^6 = t_{d_1} t_{d_2}$$

Dehn twist generators

Theorem 1.1 (Dehn,1938)

$\mathrm{Mod}(\Sigma_{g,0})$ *is generated by finitely many Dehn twists.*

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Theorem 1.2 (Lickorish, 1961)

$\mathrm{Mod}(\Sigma_{g,0})$ *is generated by* $3g-1$ *Dehn twists*
$t_{a_1}, t_{a_2}, \ldots, t_{a_g}, t_{b_1}, t_{b_2}, \ldots, t_{b_g}, t_{c_1}, t_{c_2}, \ldots, t_{c_{g-1}}$.

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Theorem 1.3 (Humphries, 1979)

$\mathrm{Mod}(\Sigma_{g,0})$ *is generated by* $2g+1$ *Dehn twists* $t_{a_1}, t_{a_2}, t_{b_1}, t_{b_2}, \ldots, t_{b_g}, t_{c_1}, t_{c_2}, \ldots, t_{c_{g-1}}$. *This is the minimum number of Dehn twists generating* $\mathrm{Mod}(\Sigma_{g,0})$.

Involution generators

Theorem 1.4 (MacCarthy-Papadopoulus, 1987)

$\mathrm{Mod}(\Sigma_{g,0})$ *is generated by infinitely many involutions.*

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Theorem 1.7 (Kassbov, 2003)

(1) $\mathrm{Mod}(\Sigma_{g,n})$ *is generated by 4 involutions.* $(g \geq 8)$

(2) $\mathrm{Mod}(\Sigma_{g,n})$ *is generated by 5 involutions.* $(g \geq 6)$

(3) $\mathrm{Mod}(\Sigma_{g,n})$ *is generated by 6 involutions.* $(g \geq 4)$

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Theorem 1.8 (Monden, 2008)

(1) $\mathrm{Mod}(\Sigma_{g,n})$ *is generated by 4 involutions.* $(g \geq 7)$
(2) $\mathrm{Mod}(\Sigma_{g,n})$ *is generated by 5 involutions.* $(g \geq 5)$

Torsion generator

Theorem 1.9 (Brendle-Farb, 2004)

When $g \geq 3$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order* $2g+2$, $4g+2$, 2.

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Theorem 1.11 (Monden, 2012)

When $g \geq 3$,

(1) $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order* 3.

(2) $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order* 4.

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Theorem 1.12 (Du, 2015)

(1) *When* $g \geq 4$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three involutions and a element of order* 3.

(2) *When* $g \geq 3$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four involutions and a element of order* 3.

Torsion generator

Theorem 1.13 (Y)

(1) *When* $g \geq 10$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by three elements of order* 6.

(2) *When* $g \geq 5$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order* 6.

Torsion generator

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(2) *When* $g \geq 5$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order* 6.

Theorem 1.14 (Lanier)

For $k \geq 5$ *and* $g \geq (k-1)(k-3)$, $\mathrm{Mod}(\Sigma_{g,0})$ *is generated by four elements of order* k. *If* k *is also a multiple of three, then only three elements of order* k *are required.*

Lantern relation

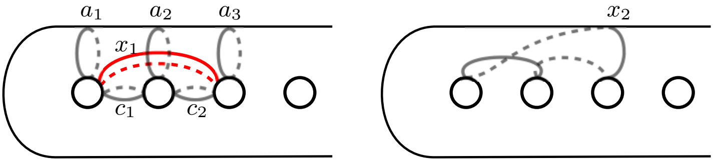
The key idea generating a Dehn twist is to use lantern relation

.

Lemma 1.3

(lantern relation) Let x_1 and x_2 be simple closed curves as shown in below. Then we have

$$t_{a_1}t_{c_1}t_{c_2}t_{a_3} = t_{x_1}t_{x_2}t_{a_2}.$$



Then rewrite lantern relation as follow,

$$t_{a_1} = (t_{x_1}t_{c_1}^{-1})(t_{x_2}t_{a_3}^{-1})(t_{a_2}t_{c_2}^{-1}).$$

Generating Dehn twist

Suppose that we can find elements of order six f and h such that $f^4(a_2) = x_1, f^2(a_2) = x_2, f^4(c_2) = c_1, f^2(c_2) = a_3$ and $h(c_2) = a_2$.
Let k be $t_{c_2} h^{-1} t_{c_2}^{-1}$. k has order six.
Then we have

$$t_{a_2} t_{c_2}^{-1} = t_{h(c_2)} t_{c_2}^{-1} = h t_{c_2} h^{-1} t_{c_2}^{-1} = hk.$$

$$t_{x_1} t_{c_1}^{-1} = t_{f^4(a_2)} t_{f^4(c_2)}^{-1} = f^4 t_{a_2} t_{c_2}^{-1} f^{-4} = f^4 hk f^{-4}.$$

$$t_{x_2} t_{a_3}^{-1} = t_{f^2(a_2)} t_{f^2(c_2)}^{-1} = f^2 t_{a_2} t_{c_2}^{-1} f^{-2} = f^2 hk f^{-2}.$$

By Lantern relation,

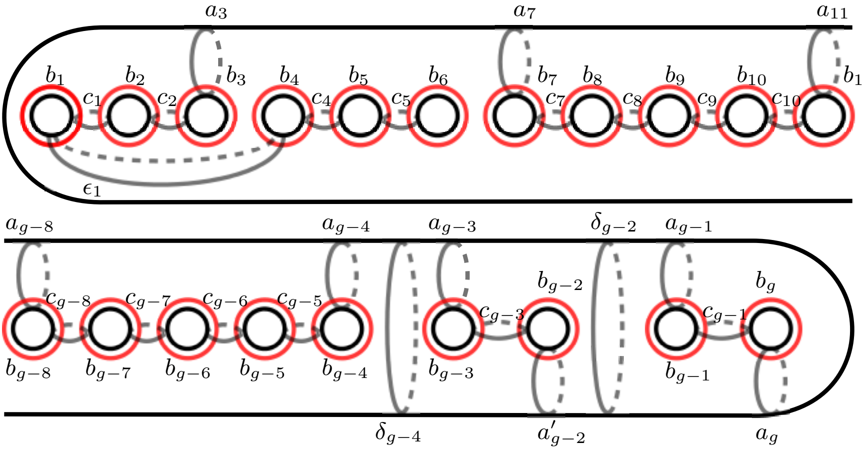
$$t_{a_1} = (f^4 hk f^{-4})(f^2 hk f^{-2})(hk).$$

Hence t_{a_1} is a product of elments of order six.

Construct elements f which has order six.

Cut the surface Σ_g along the curves

$a_3, c_1, c_2, \epsilon_1, c_4, c_5, a_{5i-3}, c_{5i-3}, c_{5i-2}, c_{5i-1}, c_{5i}, a_{5i+1}$ $(i = 2, 3, \ldots, \frac{g-5}{5})$, and δ_{g-4} as shown in below.



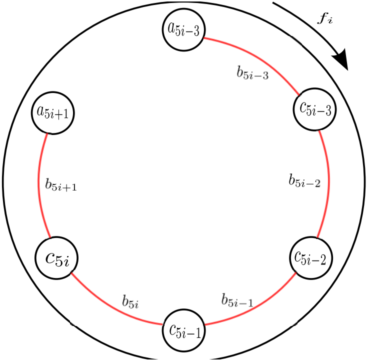
Construct element of order six I

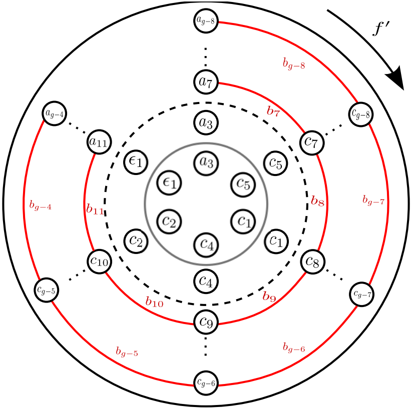
$S_1 := \Sigma_{0,\frac{6g-18}{5}}$

$S_j := \Sigma_{0,6}$ s.t. $\partial S_j = a_{5j-3} \cup c_{5j-3} \cup c_{5j-2} \cup c_{5j-1} \cup c_{5j} \cup a_{5j+1}$ $(j = 2, 3, \ldots, \frac{g-5}{5})$

$S_1' := \Sigma_{4,1}$ s.t $\partial S_1' = \delta_{g-4}$

Let $f_1', f_2, \ldots, f_{\frac{g-5}{5}}$ be $\frac{\pi}{3}$ rotation as shown in below.

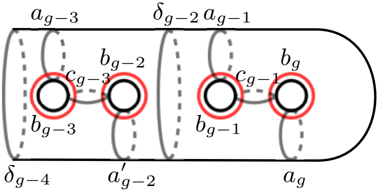




Construct element of order six I

Remark that $(f_1')^6 = t_{\delta_{g-4}}$.

$$f_1'' = (t_{a_{g-3}} t_{b_{g-3}} t_{c_{g-3}} t_{b_{g-2}} t_{a'_{g-2}})^{-1} (t_{a_{g-1}} t_{b_{g-1}} t_{c_{g-1}} t_{b_g} t_{a_g}).$$

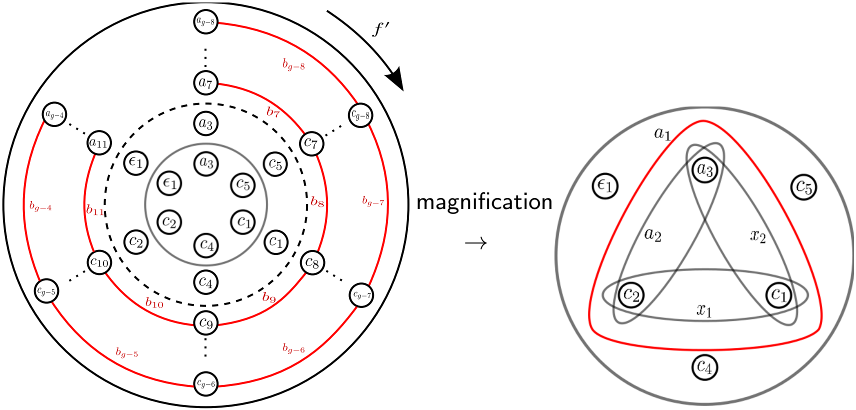


Note that $(f_1'')^6 = t_{\delta_{g-4}}^{-1}$.
$f_1', f_1'', f_2, \ldots, f_{\frac{g-5}{5}}$ define an element f of order six.

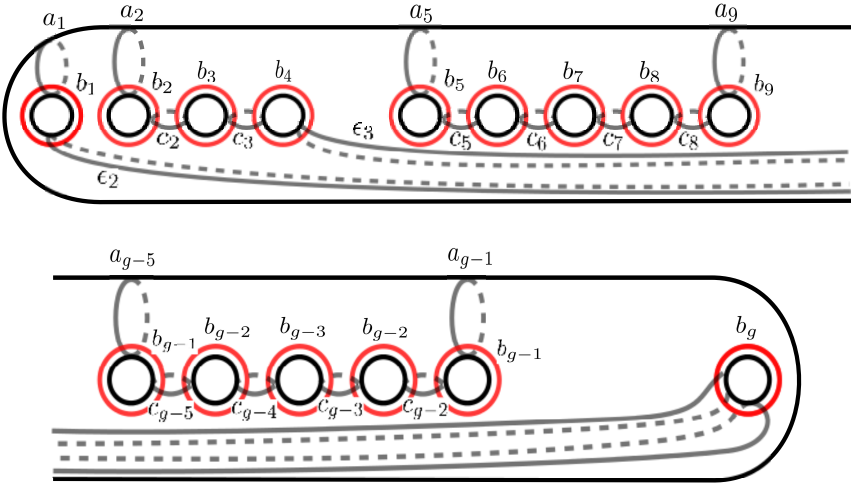
Construct element of order six I

note that f act the curves as follows.

$$f^4(a_2) = x_1, f^2(a_2) = x_2, f^4(c_2) = c_1, f^2(c_2) = a_3.$$



Construct elements h which has order six. cut the surface Σ_g along the curves $a_1, a_2, c_2, c_3, \epsilon_2, \epsilon_3, a_{5i-5}, c_{5i-5}, c_{5i-4}, c_{5i-3}, c_{5i-2}$,and a_{5i-1} $(i = 2, 3, \ldots, \frac{g}{5})$ as shown in below.

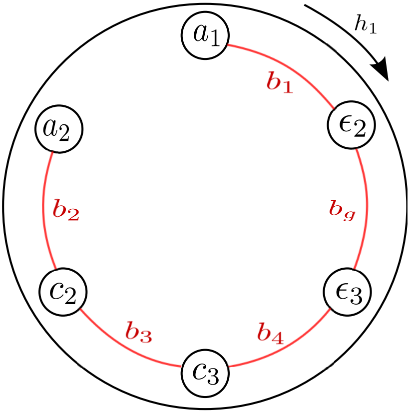


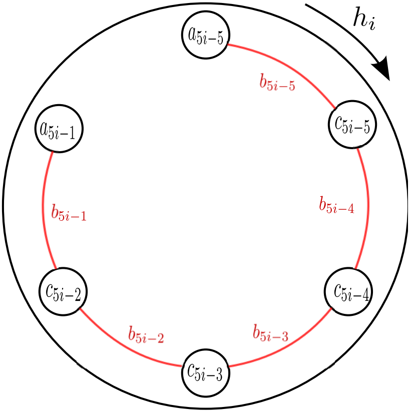
Construct element of order six II

$T_1 := \Sigma_{0, \frac{6(g-5)+12}{5}}$

$T_j := \Sigma_{0,6}$ s.t. $\partial T_j = a_{5j-3} \cup c_{5j-3} \cup c_{5j-2} \cup c_{5j-1} \cup c_{5j} \cup a_{5j+1}$ $(j = 2, 3, \dots, \frac{g-5}{5})$

Let $h_1, h_2, \dots, h_{\frac{g+5}{5}}$ be $\frac{\pi}{3}$ rotation as follows.





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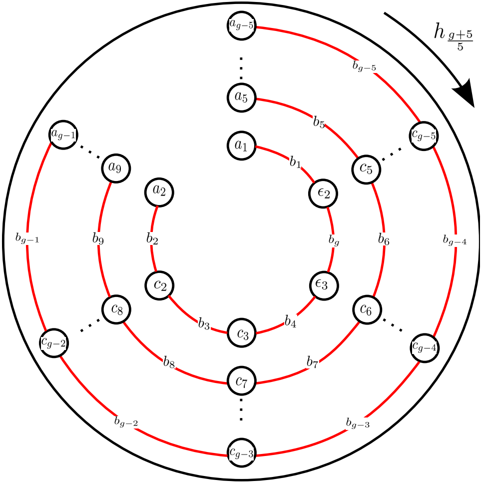


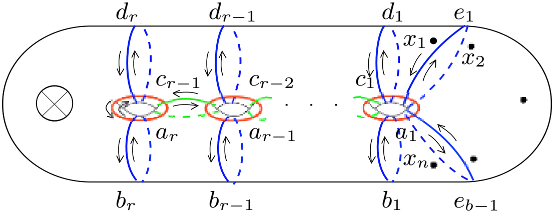
$h_1, h_2, \ldots, h_{\frac{g+5}{5}}$ define an element h of order six.

note that $h(c_2) = a_2$.

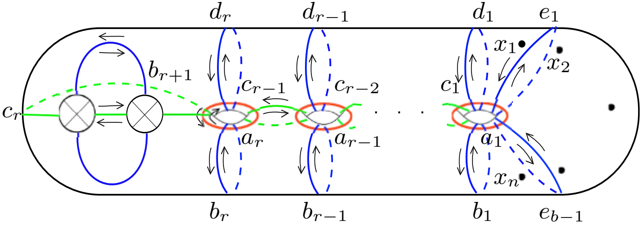
Non-orientable surface

$N_{g,n}$: a closed non-orientable surface of genus g with n punctures $P = \{x_1, x_2, \cdots, x_n\}$.
$\mathrm{Diff}(N_{g,n}) := \{f : N_{g,n} \to N_{g,n}$ |differomorphism, $f(P) = P\}$
$\mathrm{Diff}_0(N_{g,n}) := \{f \in \mathrm{Diff}(N_{g,n}) \mid f$ is isotopic to identity $\}$
$\mathrm{Mod}(N_{g,n}) := \mathrm{Diff}(N_{g,n})/\mathrm{Diff}_0(N_{g,n})$
: the mapping class group of $N_{g,n}$
$\mathrm{PMod}(N_{g,n}) := \{f \in \mathrm{Mod}(N_{g,n}) \mid f(x_i) = x_i \quad (i = 1, 2, \cdots, n)\}$
: the pure mapping class group of $N_{g,n}$
$Sym_n :=$ symmetric group on n letters
We have the exact sequence

$$1 \to \mathrm{PMod}(N_{g,n}) \to \mathrm{Mod}(N_{g,n}) \xrightarrow{\pi} Sym_n \to 1.$$



For $g = 2r + 1$**, surface** $N_{g,n}$



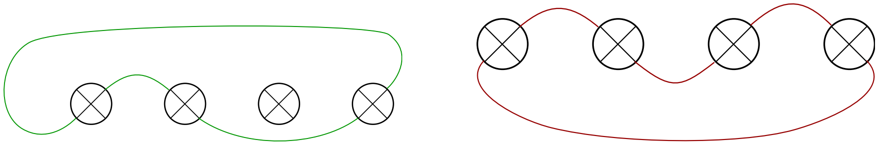
For $g = 2r + 2$**, surface** $N_{g,n}$

simple closed curve on $N_{g,n}$

c : a simple closed curve on $N_{g,n}$.

c is a two-sided $\Leftrightarrow$ the regular neighborhood of c is an annulus.

c is a one-sided $\Leftrightarrow$ the regular neighborhood of c is a Möbius band.



one and two-sided simple closed curves on $N_{g,n}$

Dehn twist of $\mathrm{Mod}(N_{g,n})$

a : two-sided simple closed curve on $N_{g,n}$.
Then we can define the Dehn twist t_a along a.

Lemma 2.1

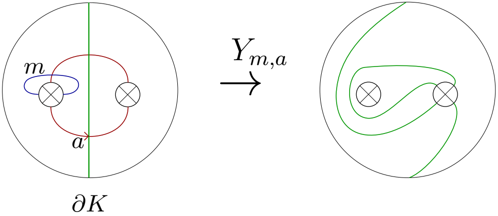
a : a two-sided simple closed curve on $N_{g,n}$.
For $f \in \mathrm{Mod}(N_{g,n})$, $t^{\epsilon}_{f(a)} = f t_a f^{-1}$
Where, $N_a :=$ the regular neighborhood of a.
$f \mid N_a$ is orientation preserving $\Rightarrow \epsilon = 1$.
$f \mid N_a$ is orientation reversing $\Rightarrow \epsilon = -1$.

m : one-sided simple closed curve on $N_{g,n}$

a : two-sided simple closed curve on $N_{g,n}$

$K :=$ the regular neighborhood of $m \cup a$ ($\cong$ (the Klein bottle with one hole))

$Y_{m,a} :=$ the Y-homeomorphism.



Y-homeomorphism on K

note that $Y_{m,a}^2 = t_{\partial K}$.

Lemma 2.2

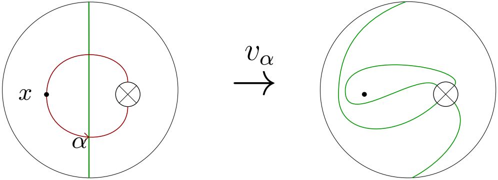
(1) $Y_{m^{-1},a} = Y_{m,a}$.

(2) $Y_{m,a^{-1}} = Y_{m,a}^{-1}$.

(3) *For* $f \in \mathrm{Mod}(N_{g,n})$, $fY_{m,a}f^{-1} = Y_{f(m),f(a)}$.

Puncture slide

α : one-sided simple closed curve on $N_{g,n}$, based at the puncture x
M := the regular neighborhood of α ($\cong$ Möbius band with one puncture)



puncture slide along α on M

Lemma 2.3

For $f \in \mathrm{Mod}(N_{g,n})$, $f v_\alpha f^{-1}$ is the puncture slide of $f(x)$ along $f(\alpha)$.

Generator for $\mathrm{Mod}(N_{g,n})$

Theorem 2.1 (Lickorish, 1963)

(1) $\mathrm{Mod}(N_{g,0})$ *is generated by Dehn twists and Y-homeomorphism.*

(2) $\mathrm{Mod}(N_{g,0})$ *is not generated by Dehn twists.*

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Theorem 2.2 (Chillingworth, 1969)

$\mathrm{Mod}(N_{g,0})$ *is generated by finite generating set.* $\quad (g \geq 3)$

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Theorem 2.3 (Korkmaz, 2002)

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$\mathrm{Mod}(N_{g,n})$ *is generated by finite generating set* $(g \geq 3)$.

Theorem 2.4 (Szepietowski, 2013 , Hirose, 2016)

$\mathrm{Mod}(N_{g,0})$ *is generated by* g *Dehn twists and a Y-homeomorphism. Moreover, this generator set is minimal generator set by Dehn twists and Y-homeomorphisms.*

Involution generator for $\mathrm{Mod}(N_{g,b})$

Theorem 2.5 (Szepietowski, 2004)

For $g \geq 1$, $\mathrm{Mod}(N_{g,n})$ is generated by involutions.

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Theorem 2.6 (Szepietowski, 2006)

For $g \geq 4$, $\mathrm{Mod}(N_{g,0})$ is generated by 4 involutions.

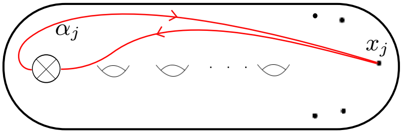
Involution generator for $\mathrm{Mod}(N_{g,n})$

Theorem 3.1 (Y)

$\mathrm{Mod}(N_{g,n})$ is generated by 8 involutions. ($g \geq 13$ and g is odd)
$\mathrm{Mod}(N_{g,n})$ is generated by 11 involutions. ($g \geq 14$ and g is even)

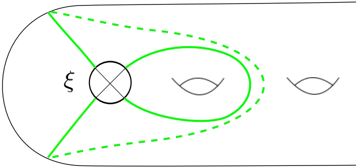
Suppose that $g = 2r + 1$, $r = 2k$ and $n = 2l + 1$.

$v_j :=$ the puncture slide of x_j along α_j.



Generator for $\mathrm{PMod}(N_{g,b})$

y := the Y-homeomorphism s.t. $y^2 = t_\xi$



$S := \{a_1, a_2, \cdots, a_r, b_1, b_2, c_1, c_2, \cdots, c_{r-1}, d_1, d_2, e_1, e_2, \cdots, e_n - 1\}$

Theorem 3.2 (Korkmaz, 2002)

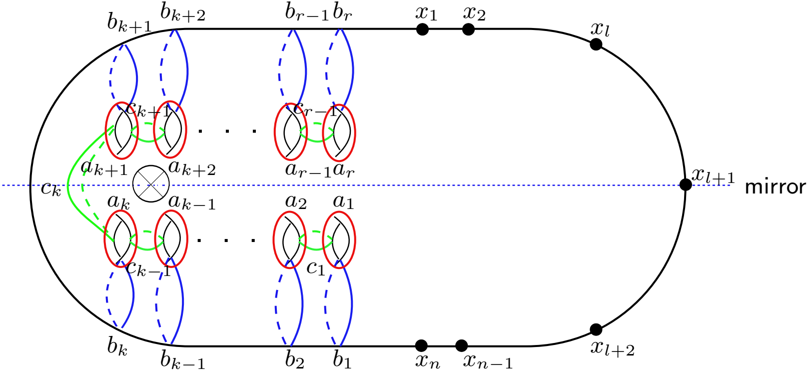
$\mathrm{PMod}(N_{g,n})$ *is generated by following elements.*

(1) t_l *for* $l \in S$.

(2) v_j *for* $1 \leq j \leq n$.

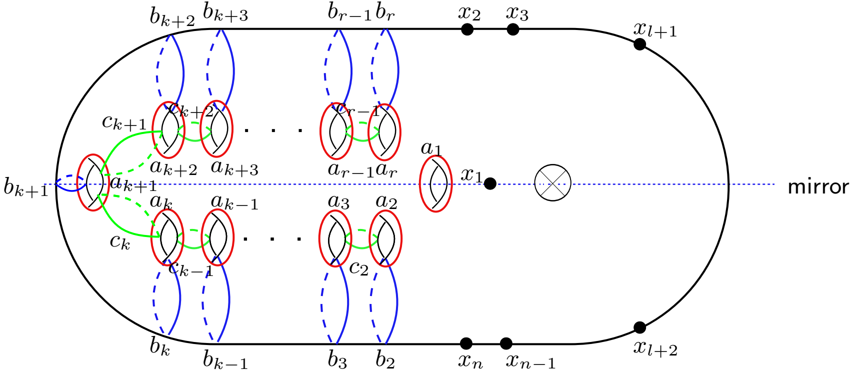
(3) y.

the next figure gives the involution σ.



The mirror image σ

the next figure gives the involution τ.

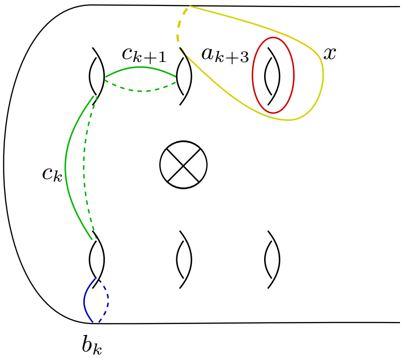


The mirror image τ

involution I

We will construct the third involution.

Cut the surface along $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$.

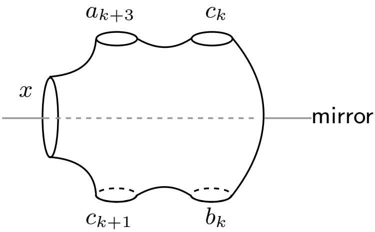


S_1 := the five holed sphere bounded by $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$.

S_2 := $N_{g-8,b}$ bounded by $a_{k+3} \cup b_k \cup c_k \cup c_{k+1} \cup x$.

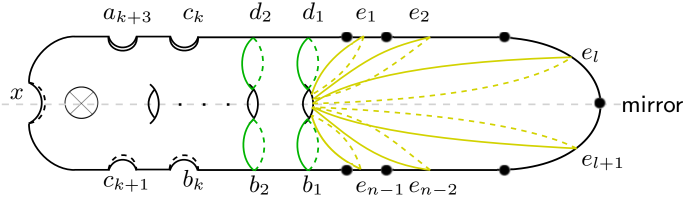
involution I

the next figure gives the involution $\overline{I}$ on S_1.



The mirror image $\overline{I}$ on S_1

the next figure gives the involution $\widetilde{I}$ on S_2.



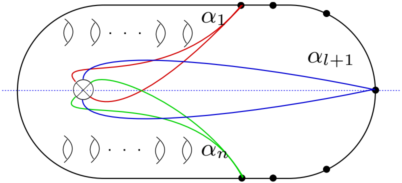
The mirror image $\widetilde{I}$ on S_2

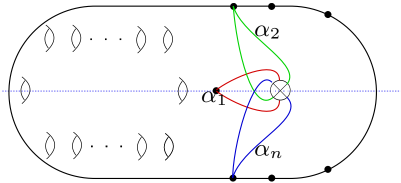
$\overline{I}$ and $\widetilde{I}$ define the involution I on $N_{g,n}$.

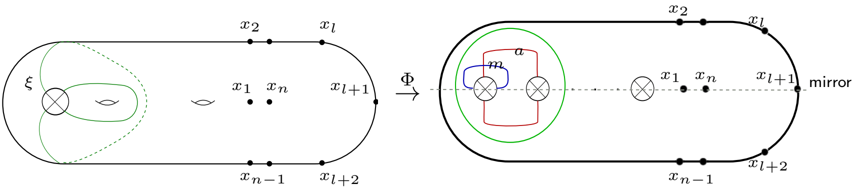
Generating Dehn twist and puncture slide

$\rho_1 := \tau t_{a_1}$.
Since $\tau t_{a_1} \tau = t_{a_1}^{-1}$,
$\rho_1^2 = \underline{\tau t_{a_1} \tau} t_{a_1} = t_{a_1}^{-1} t_{a_1} = id$.
$\therefore \rho_1$ is involution.

$\rho_2 := \tau v_1$.
Since $\tau(\alpha_1) = \alpha_1^{-1}$,
$\tau v_1 \tau$ is the puncture slide of puncture $\tau(x_1) = x_1$ along $\tau(\alpha_1) = \alpha_1^{-1}$.
$\therefore \tau v_1 \tau = v_1^{-1}$
$\rho_2^2 = \underline{\tau v_1 \tau} v_1 = v_1^{-1} v_1 = id$.
$\therefore \rho_2$ is the involution.







diffeo $\Phi : N_{g,b} \to N_{g,b}$ s.t. $\Phi y \Phi^{-1} = Y_{m,a}$.

$w :=$ the reflection of the right model in above figure.

$w(m) = m^{-1}$ and $w(a) = a^{-1}$.

$wY_{m,a}w = Y_{w(m),w(a)} = Y_{m^{-1},a^{-1}} = Y_{m,a}^{-1}$.

$W := \Phi^{-1} w \Phi$.

$\rho_3 := Wy$.

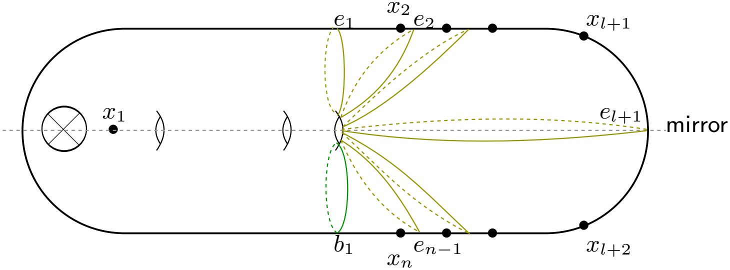
$WyW = \Phi^{-1} w \underline{\Phi y \Phi^{-1}} w \Phi = \Phi^{-1} \underline{w \underline{Y_{m,a}} w} \Phi = \Phi^{-1} \underline{Y_{m,a}^{-1}} \Phi = y^{-1}$.

Hence, $\rho_3^2 = \underline{WyW}y = \underline{y^{-1}}y = id$.

$y = W \cdot Wy$.

Involution J

the next figure gives the involution J.



The mirror image J on $N_{g,n}$

Generator for Sym_n

Lemma 3.1

Sym_n is generated by

$r_1 = (1, n)(2, n-1)\cdots(l, l+2)(l+1)$

$r_2 = (2, n)(3, n-1)\cdots(l+1, l+2)(1)$

$r_3 = (2, n-1)(3, n-2)\cdots(l, l+2)(1)(l+1)(n).$

$\pi(\sigma) = (1, n)(2, n-1)\cdots(l, l+2)(l+1).$

$\pi(\tau) = (2, n)(3, n-1)\cdots(l+1, l+2)(1).$

$\pi(W) = (2, n-1)(3, n-2)\cdots(l, l+2)(1)(l+1)(n).$

Coxeter group

$G := \langle g_1, g_2, \ldots, g_n \mid (g_i g_j)^{m_{ij}} = 1 \rangle$
we call the group G Coxter group.
where $m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$.
$m_{ij} = \infty$ means no relation of the form $(g_i g_j)^{m_{ij}}$.

Cor 3.1

For $g \geq 13$ and g is odd,
$\mathrm{Mod}(N_{g,n})$ *can be realized as a quotient of a Coxter group on* 8 *generators.*
For $g \geq 14$ and g is even,
$\mathrm{Mod}(N_{g,n})$ *can be realized as a quotient of a Coxter group on* 11 *generators.*

Thank you for your attention.