ハンドル体結び目とpartially multiplicative biquandleについて

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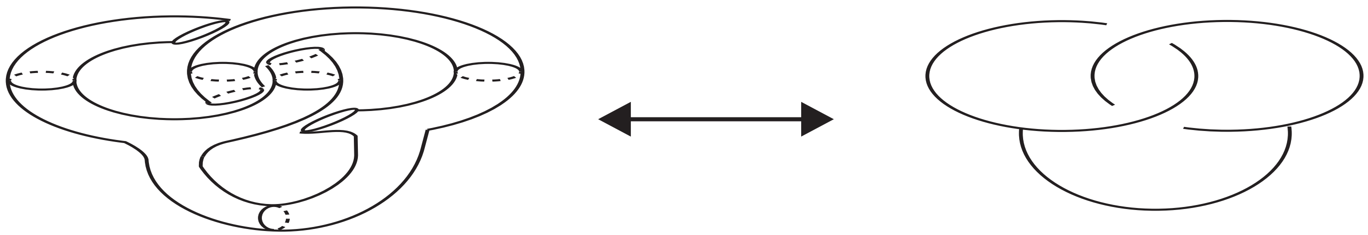
Handlebody-knot

handlebody-knot $\Leftrightarrow$ an embedded handlebody in $\mathbb{S}^3$

$H \cong H' \Leftrightarrow \exists$ ori. pres. homeo. h of $\mathbb{S}^3$ with $h(H) = H'$.

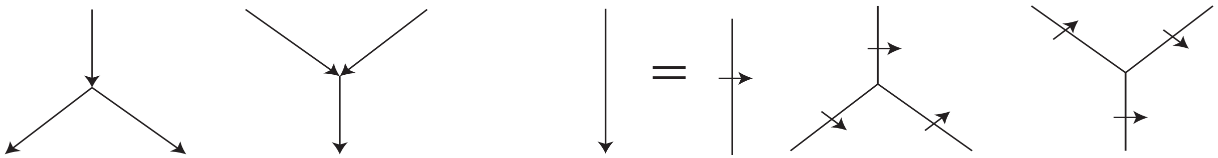
handlebody-knot H $\underset{\text{regular nbd.}}{\overset{\text{spine}}{\rightleftarrows}}$ trivalent spatial graph G



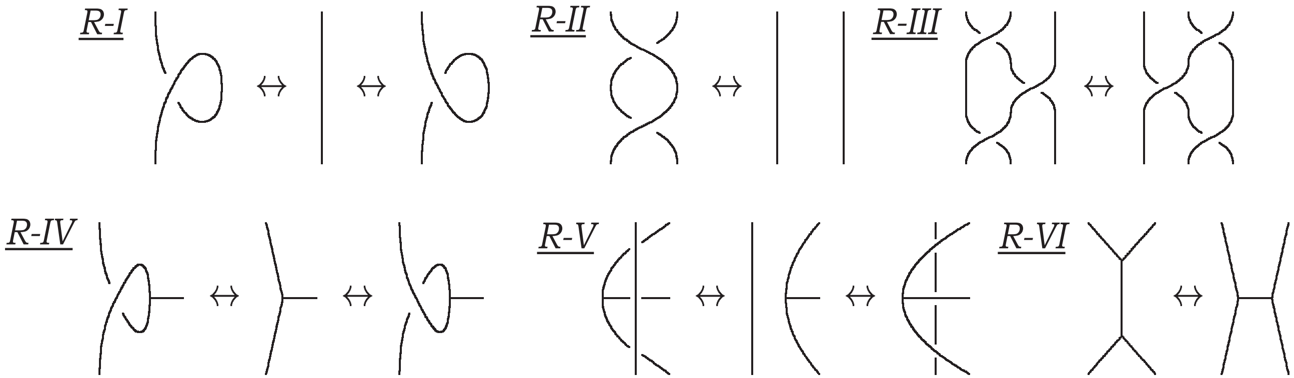
genus one handlebody-knot $\Longleftrightarrow$ classical knot

diagram of H $\overset{\text{def.}}{\Longleftrightarrow}$ diagram of G

Y-orientation for a trivalent graph (diagram)



R-moves for handlebody-knots



Theorem 1 (Ishii). *H_i:a handlebody-knot represented by a (Y-ori.) diagram D_i for $i = 1, 2$. Then $H_1 \cong H_2 \Leftrightarrow D_1, D_2$ are related by a finite number of R-moves (preserving Y-orientations).*

- Coloring via a quandle on a diagram with a flow (Ishii-I)

- Coloring via a G-family of quandles (Ishii-I-Jang-Oshiro)

- Coloring via a multiple conjugation quandle (Ishii)

- Coloring via a partially muitiplicative biquandle (Ishii-Nelson, Ishii-I-Kamada-Kim-Matsuzaki-Oshiro)

Quandle $X \neq \emptyset$ is a quandle with a map $* : X \times X \to X$ satisfying the following axioms:

1. $\forall x \in X, x * x = x$.
2. $\forall x \in X$, $*x : X \to X$ with $a \mapsto a * x$ is bijective.
3. $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Ex. • A group G is a conjugation quandle with $g * h = h^{-1}gh$.
• $\Lambda := \mathbb{Z}[t^{-1}, t]$. A Λ-module M is an Aleander quandle with $x * y = xt + y(1 - t)$.
• R_n is the dihedral quandle when $R_n = \mathbf{Z}_n$ with $x * y = 2y - x$.

A quandle X is of type m if

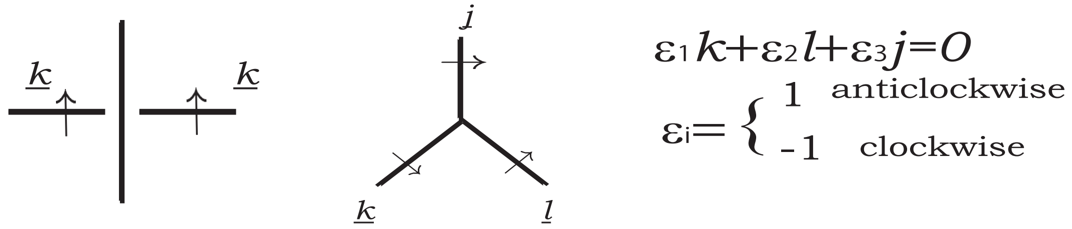
$$m = \min\{l > 0 \mid (x *^l y :=)(\cdots(x * \overbrace{y) * y) \cdots) * y}^{l} = x \text{ for } \forall x, y \in X\}.$$

If $\nexists n$, then we call it of type ∞. R_n is of type 2.

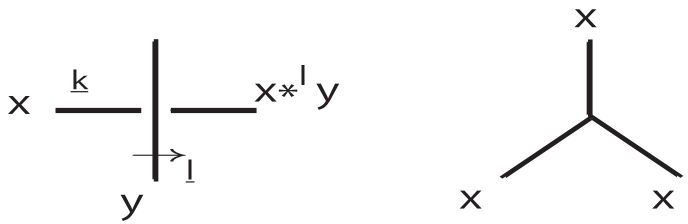
Coloring via a quandle on a diagram with a flow

D: a Y-ori. diagram, $\mathcal{A}(D)$:={arcs of D}. A : abelian group.

A map $\varphi : \mathcal{A}(D) \to A$ is an A-flow of D if



X: a quandle of type n, a map $C : \mathcal{A}(D) \to X$ is an X-coloring of D with $\mathbb{Z}_n$-flow if



$(\mathbb{Z}_\infty = \mathbb{Z})$

Coloring via a G-family of quandles

G is a group with identity e. $X \neq \emptyset$ is a G-family of quandles with a map $*^g : X \times X \to X$ for each $g \in G$ satisfying the following axioms:

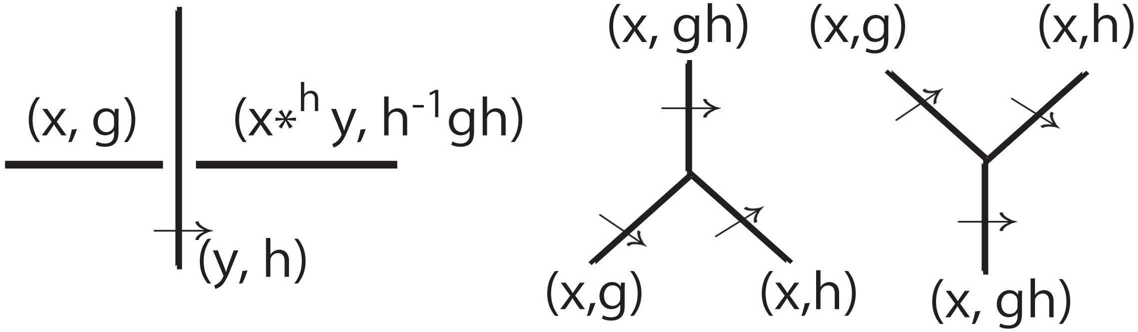
1. $\forall x \in X, \forall g \in G, x *^g x = x$.
2. $\forall x, y \in X, \forall g, h \in G$, $x *^{gh} y = (x *^g y) *^h y$ and $x *^e y = x$.
3. $\forall x, y, z \in X, \forall g, h \in G$, $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z)$.

Ex. • X:a quandle of type n. Then X is a $\mathbb{Z}_n$-family of quandles with $x *^l y := (\cdots(x * \overbrace{y) * y) \cdots) * y}^{l}$ for $\forall x, y \in X, l \in \mathbb{Z}_n$. (Ishii-I version)

• R: a ring, G: a group with identity e, X: a right $R[G]$-module. Then X is a G-family of quandles with $x *^g y := xg + y(e - g)$ for $\forall x, y \in X, g \in G$, which we call it a G-family of Alexander quandles.

If X is an Alexander quandle of type n, then $x *^l y := xt^l + y(1 - t^l)$. X is an example in both cases when $R = G = \mathbb{Z}$ and $R[G] = \Lambda$.

D: a Y-ori. diagram. X: a G-family of quandles, a map $C : \mathcal{A}(D) \to X \times G$ is an X-coloring of D if



Re. X: a G-family of quandles.

- $\forall g \in G$, X is a quandle with $*^g$.
- $X \times G$ is a quandle with $(x, g) * (y, h) = (x *^h y, h^{-1}gh)$.

Coloring via a multiple conjugation quandle (Ishii)

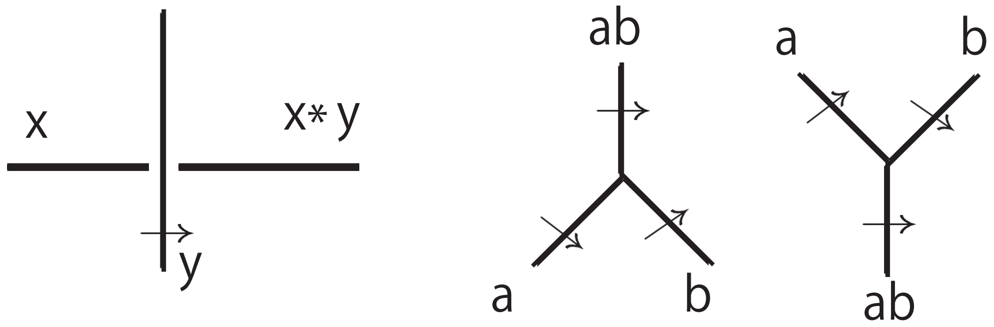
For groups $G_\lambda (\lambda \in \Lambda)$, $X = \coprod_{\lambda \in \Lambda} G_\lambda$ is a multiple conjugation quandle with a map $* : X \times X \to X$ satisfying the following axioms (If $a \in G_\lambda$, then $G_a := G_\lambda$.):

1. $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.
2. $\forall a, x \in X$, $*x : G_a \to G_{a*x}$ is a group homomorphism.
3. $\forall a, b \in G_\lambda$, $\forall x \in X$, $x * ab = (x * a) * b$, $x * e_\lambda = x$, $a * b = b^{-1}ab$.

Re. • A multiple conjugation quandle is a quandle.

• X: a G-family of quandles. $G_x := \{x\} \times G (\subset X \times G)$. Then $\coprod_{x \in X} G_x$ is a multiple conjugation quandle with $(x, g) * (y, h) = (x *^h y, h^{-1}gh)$, $(x, g)(x, h) = (x, gh)$ for $\forall x, y \in X$, $\forall g, h \in G_x$.

D: a Y-ori. diagram. X: a multiple conjugation quandle, a map $C : \mathcal{A}(D) \to X$ is an X-coloring of D if for $\forall x, y \in X$, $\forall a, b \in G_\lambda$,



Re. A multiple conjugation quandle is the universal quandle with partial multiplications : $P(\subset X \times X) \to X$ to define coloring invariants under the condition $(\star)$ that

$$\forall a \quad \exists b \quad \text{such that} \quad (a, b) \in P \quad (\overset{def}{\Leftrightarrow}: a \sim b).$$

If $\not\exists$ such b for a, then a cannot entry as a color of any arc in any diagram when genus ≥ 2.

biquandle $X \neq \emptyset$ is a biquandle with maps $\underline{*}, \overline{*} : X \times X \to X$ satisfying the following axioms:

1. $\forall x \in X, x \underline{*} x = x \overline{*} x$.
2. $\forall x \in X$, $\underline{*} x : X \to X$ with $a \mapsto a \underline{*} x$ is bijective.
   $\forall x \in X$, $\overline{*} x : X \to X$ with $a \mapsto a \overline{*} x$ is bijective.
   $S : X \times X \to X \times X$ with $S(x, y) = (y \overline{*} x, x \underline{*} y)$ is bijective.
3. $$\forall x, y, z \in X, \begin{aligned} (x \underline{*} y) \underline{*} (z \underline{*} y) &= (x \underline{*} z) \underline{*} (y \overline{*} z), \\ (x \underline{*} y) \overline{*} (z \underline{*} y) &= (x \overline{*} z) \underline{*} (y \overline{*} z), \\ (x \overline{*} y) \overline{*} (z \overline{*} y) &= (x \overline{*} z) \overline{*} (y \underline{*} z). \end{aligned}$$

Ex. • R: an ring. An $R[s^{\pm 1}, t^{\pm 1}]$-module X is an Aleander biquandle with $x \underline{*} y = tx + (s - t)y, x \overline{*} y = sx$.

• A gruop G is a biquandle with $\underline{*}, \overline{*}$ given in each of followings:

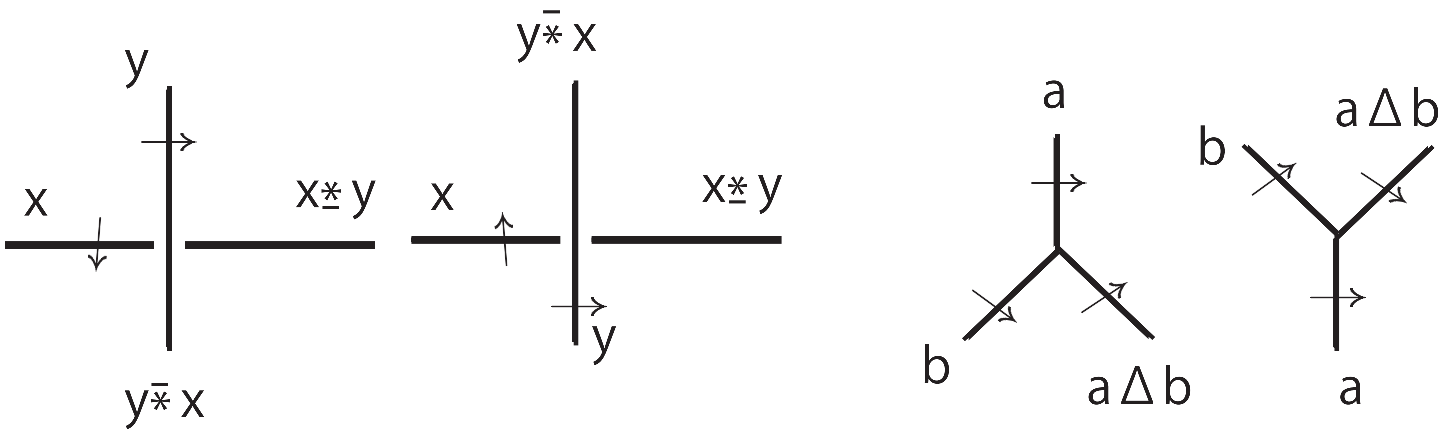
(1) $g \underline{*} h = g \overline{*} h = g^{-1}$.

(2) $g \underline{*} h = h^{-1} g h^{-1}, g \overline{*} h = g^{-1}$.

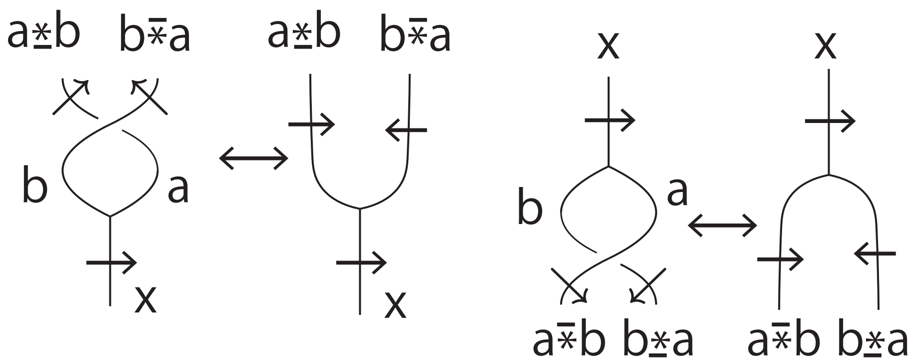
(3) $g \underline{*} h = h^{-2} g, g \overline{*} h = h^{-1} g^{-1} h$.

We would like to a universal biquandle with partial multiplications to define coloring invariants under the condition $(\star)$ that

$$\forall a \quad \exists b \;\text{ such that }\; (a, b) \in P \quad (\overset{def}{\Leftrightarrow}: a \sim b).$$



Primitive conditions



(R4) $\forall a, b, x \in X$,

$$a \sim b, x = a \triangle b \Leftrightarrow a \underline{*} b \sim x, (a \underline{*} b) \triangle x = b \overline{*} a,$$

$$a \sim b, x = a \triangle b \Leftrightarrow a \overline{*} b \sim x, (a \overline{*} b) \triangle x = b \underline{*} a.$$

(R5) $\forall a, b, x \in X$,

$$a \sim b \Leftrightarrow a \underline{*} x \sim b \underline{*} x \;\Rightarrow (x \overline{*} b) \overline{*} (a \triangle b) = x \overline{*} a, (a \triangle b) \underline{*} (x \overline{*} b) = (a \underline{*} x) \triangle (b \underline{*} x),$$
$$a \sim b \Leftrightarrow a \overline{*} x \sim b \overline{*} x \;\Rightarrow (x \underline{*} b) \underline{*} (a \triangle b) = x \underline{*} a, (a \triangle b) \overline{*} (x \underline{*} b) = (a \overline{*} x) \triangle (b \overline{*} x).$$

(R6) $\forall a, b, c, x \in X$,

$$a \sim b, b \sim c, x = b \triangle c \Rightarrow a \sim c, a \triangle c \sim x, (a \triangle c) \triangle x = a \triangle b,$$
$$\exists ! b \in X \text{ s.t. } a \sim b, b \sim c, x = b \triangle c, (a \triangle c) \triangle x = a \triangle b \Leftarrow a \sim c, a \triangle c \sim x,$$
$$a \sim b, a \sim c, x = a \triangle c \Rightarrow b \sim c, x \sim b \triangle c, x \triangle (b \triangle c) = a \triangle b,$$
$$\exists ! a \in X \text{ s.t. } a \sim b, a \sim c, x = a \triangle c, x \triangle (b \triangle c) = a \triangle b \Leftarrow b \sim c, x \sim b \triangle c.$$

Re. • $\sim$ is an equivalence relation.
• If $\not\exists$ b for a such that $a \sim b$, then a cannot entry as a color of any arc in any diagram when genus ≥ 2.

triangle biquandle A triangle biquandle $X = \coprod_{\lambda \in \Lambda} G_\lambda$ (G_λ: a set) is with $\underline{*}, \overline{*} : X \times X \to X$ and $\triangle : \coprod_{\lambda \in \Lambda} G_\lambda^2 \to X$ satisfying the followings, where $a \triangle b := \triangle(a, b)$. (If $a \in G_\lambda$, then $G_a := G_\lambda$.)

(1) $(X, \underline{*}, \overline{*})$ is a biquandle.

(2) $\forall a \in X$, $\triangle a : G_a \to G_{a \triangle a}$ with $x \mapsto x \triangle a$ is a bijection.

$\forall a, x \in X$, $\underline{*} x : G_a \to G_{a \underline{*} x}$ and $\overline{*} x : G_a \to G_{a \overline{*} x}$ are bijections.

(3) $\forall a, b \in G_\lambda$,

$$G_{a \underline{*} b} = G_{a \overline{*} b} = G_{a \triangle b}, (a \underline{*} b) \triangle (a \triangle b) = b \overline{*} a, (a \overline{*} b) \triangle (a \triangle b) = b \underline{*} a.$$

(4) $\forall a, b \in G_\lambda, \forall x \in X$,

$$(a \triangle b) \underline{*} (x \overline{*} b) = (a \underline{*} x) \triangle (b \underline{*} x), (a \triangle b) \overline{*} (x \underline{*} b) = (a \overline{*} x) \triangle (b \overline{*} x),$$
$$(x \underline{*} b) \underline{*} (a \triangle b) = x \underline{*} a, (x \overline{*} b) \overline{*} (a \triangle b) = x \overline{*} a.$$

(5) $\forall a, b, c \in G_\lambda$, $(a \triangle c) \triangle (b \triangle c) = a \triangle b$.

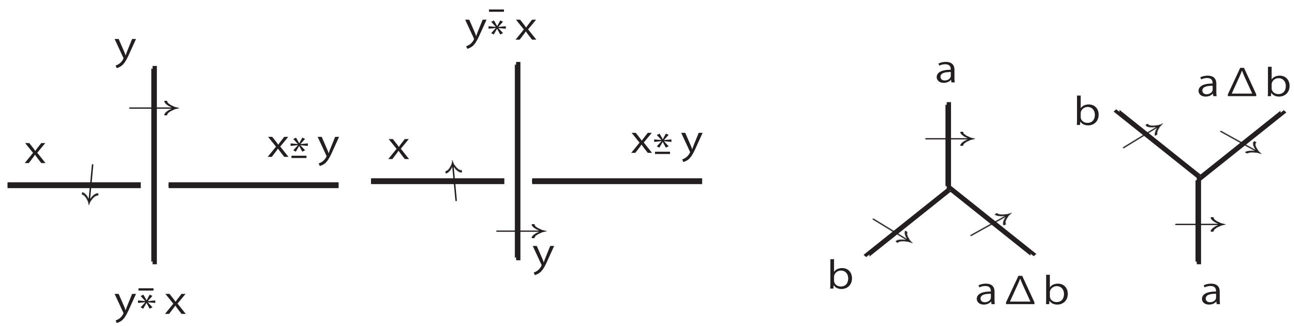
partially multiplicative biquandle A partially multiplicative biquandle is a biquandle $(X, \underline{*}, \overline{*})$ with $X = \coprod_{\lambda \in \Lambda} G_\lambda$ (G_λ: a group) satisfying the followins. (If $a \in G_\lambda$, then $G_a := G_\lambda$.)
(1) $\forall a, x \in X$, $\underline{*}x : G_a \to G_{a \underline{*} x}$ and $\overline{*}x : G_a \to G_{a \overline{*} x}$ are group homo.
(2) $\forall a, b \in G_\lambda, \forall x \in X$,

$$x \,\underline{*}\, ab = (x \,\underline{*}\, a) \,\underline{*}\, (b \,\overline{*}\, a), x \,\overline{*}\, ab = (x \,\overline{*}\, a) \,\overline{*}\, (b \,\overline{*}\, a),$$
$$a^{-1} b \,\overline{*}\, a = ba^{-1} \,\underline{*}\, a.$$

Proposition 2. *A partially multiplicative biquandle $X = \coprod_{\lambda \in \Lambda} G_\lambda$ with $a \triangle b := b^{-1} a \,\overline{*}\, b$ for $a, b \in G_\lambda$ is a triangle biquandle.*
Proposition 3. *A triangle biquandle $X = \coprod_{\lambda \in \Lambda} G_\lambda$ with $ab := b \overline{*} a \triangle^{-1} a$ for $a, b \in G_\lambda$ is a partially multiplicative biquandle. ($e_\lambda := a \triangle a \,\underline{*}^{-1}\, a = a \triangle a \,\overline{*}^{-1}\, a \in G_\lambda, a^{-1} := a \triangle a \,\underline{*}^{-1}\, a \triangle a \,\underline{*}^{-1}\, a = a \triangle a \,\overline{*}^{-1}\, a \triangle a \,\overline{*}^{-1}\, a \in G_\lambda$ for $a, b \in G_\lambda$.)*

Let $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ be a triangle biquandle, or a partially multiplicative biquandle with $a \triangle b := b^{-1} a \,\overline{*}\, b$.

D: a Y-ori. diagram, $\mathcal{SA}(D)$:=\{semi-arcs of D\}. A map $C : \mathcal{SA}(D) \to X$ is an X-coloring of D if



$\mathrm{Col}_X(D)$:= \{X-colorings of D\}.

Theorem 4. $D \underset{Y-ori.R\text{-}moves}{\longleftrightarrow} D' \implies \mathrm{Col}_X(D) \underset{1:1}{\longleftrightarrow} \mathrm{Col}_X(D')$.

Proposition 5. *An* $X = \coprod_{\lambda \in \Lambda} G_\lambda$ *(*G_λ*: a group) with* $\underline{*}, \overline{*} : X \times X \to X$ *is a partially multiplicative biquandle if and only if*

1. $\forall x, y, z \in X$*,*

$$(x \,\underline{*}\, y) \,\underline{*}\, (z \,\underline{*}\, y) = (x \,\underline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),$$
$$(x \,\underline{*}\, y) \,\overline{*}\, (z \,\underline{*}\, y) = (x \,\overline{*}\, z) \,\underline{*}\, (y \,\overline{*}\, z),$$
$$(x \,\overline{*}\, y) \,\overline{*}\, (z \,\overline{*}\, y) = (x \,\overline{*}\, z) \,\overline{*}\, (y \,\underline{*}\, z).$$

2. $\forall a, x \in X$*,* $\underline{*}x : G_a \to G_{a \underline{*} x}$ *and* $\overline{*}x : G_a \to G_{a \overline{*} x}$ *are group homo.*

3. $\forall a, b \in G_\lambda, \forall x \in X$*,*

$$x \,\underline{*}\, ab = (x \,\underline{*}\, a) \,\underline{*}\, (b \,\overline{*}\, a), \quad x \,\underline{*}\, e_\lambda = x,$$
$$x \,\overline{*}\, ab = (x \,\overline{*}\, a) \,\overline{*}\, (b \,\overline{*}\, a), \quad x \,\overline{*}\, e_\lambda = x,$$
$$a^{-1} b \,\overline{*}\, a = ba^{-1} \,\underline{*}\, a.$$

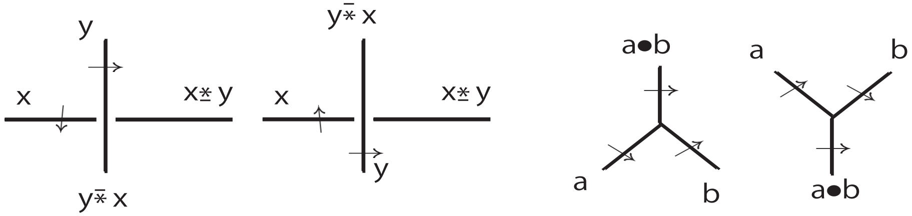
<u>Re.</u> For a multiple conjugation qunadle $(X, *)$, we can define a partially multiplicative biquandle $(X, \underline{*}, \overline{*})$ by $x \,\underline{*}\, y := x * y$ and $x \,\overline{*}\, y := x$.

For groups $G_\lambda (\lambda \in \Lambda)$, $X = \coprod_{\lambda \in \Lambda} G_\lambda$ is a multiple conjugation quandle with a map $* : X \times X \to X$ satisfying the following axioms:

1. $\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.
2. $\forall a, x \in X$, $*x : G_a \to G_{a*x}$ is a group homomorphism.
3. $\forall a, b \in G_\lambda$, $\forall x \in X$, $x * ab = (x * a) * b$, $x * e_\lambda = x$, $a * b = b^{-1}ab$.

<u>partially multiplicative biquandle (Ishii-Nelson version)</u> A partially multiplicative biquandle in the sence of [IN] is a biquandle X with $\widetilde{P} \subset X \times X$ and a map $\bullet : \widetilde{P} \to X$ satisfying the followings, where $a \bullet b := \bullet(a, b)$.

1. $x \mapsto a \bullet x$, $x \mapsto x \bullet b$ are injective.
2. $(a, b \,\underline{*}\, a) \in \widetilde{P} \Leftrightarrow (b, a \,\overline{*}\, b) \in \widetilde{P} \Rightarrow a \bullet (b \,\underline{*}\, a) = b \bullet (a \,\overline{*}\, b)$.
3. $(a, b) \in \widetilde{P} \Leftrightarrow (a \,\underline{*}\, x, b \,\underline{*}\, (x \,\overline{*}\, a)) \in \widetilde{P} \Leftrightarrow (a \,\overline{*}\, x, b \,\overline{*}\, (x \,\underline{*}\, a)) \in \widetilde{P} \Rightarrow$
$x \,\underline{*}\, (a \bullet b) = (x \,\underline{*}\, a) \,\underline{*}\, b, (a \bullet b) \,\underline{*}\, x = (a \,\underline{*}\, x) \bullet (b \,\underline{*}\, (x \,\overline{*}\, a))$,
$x \,\overline{*}\, (a \bullet b) = (x \,\overline{*}\, a) \,\overline{*}\, b, (a \bullet b) \,\overline{*}\, x = (a \,\overline{*}\, x) \bullet (b \,\overline{*}\, (x \,\underline{*}\, a))$.
4. $(a, b), (a \bullet b, c) \in \widetilde{P} \Leftrightarrow (b, c), (a, b \bullet c) \in \widetilde{P} \Rightarrow (a \bullet b) \bullet c = a \bullet (b \bullet c)$.
5. $(a, b), (c, d) \in \widetilde{P}, a \bullet b = c \bullet d \Leftrightarrow \exists e \in X$ such that $(a, e), (e, d) \in \widetilde{P}, a \bullet e = c, e \bullet d = b$.



G-family of biquandles G is a group with identity e. $X \neq \emptyset$ is a G-family of biquandles with a map $\underline{*}^g, \overline{*}^g : X \times X \to X$ $(g \in G)$ satisfying the following axioms:

1. $\forall x, y, z \in X, \forall g, h \in G$,

$$(x \,\underline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\underline{*}^h\, z) \,\underline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z),$$
$$(x \,\overline{*}^g\, y) \,\underline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\underline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z),$$
$$(x \,\overline{*}^g\, y) \,\overline{*}^h\, (z \,\overline{*}^g\, y) = (x \,\overline{*}^h\, z) \,\overline{*}^{h^{-1}gh}\, (y \,\underline{*}^h\, z).$$

2. $\forall x, y \in X, \forall g, h \in G$,

$$x \,\underline{*}^{gh}\, y = (x \,\underline{*}^g\, y) \,\underline{*}^h\, (y \,\underline{*}^g\, y), \quad x \,\underline{*}^e\, y = x,$$
$$x \,\overline{*}^{gh}\, y = (x \,\overline{*}^g\, y) \,\overline{*}^h\, (y \,\overline{*}^g\, y), \quad x \,\overline{*}^e\, y = x,$$

3. $\forall x \in X, \forall g \in G$,

$$x \,\underline{*}^g\, x = x \,\overline{*}^g\, x.$$

Type. For a biquandle $(X, \underline{*}, \overline{*})$, we define $\underline{*}^{[n]}, \overline{*}^{[n]} : X \times X \to X$ $(n \in \mathbb{Z})$ by the equalities

$$a \,\underline{*}^{[0]}\, b = a, \quad a \,\underline{*}^{[1]}\, b = a \,\underline{*}\, b, \quad a \,\underline{*}^{[i+j]}\, b = (a \,\underline{*}^{[i]}\, b) \,\underline{*}^{[j]}\, (b \,\underline{*}^{[i]}\, b), \quad (1)$$
$$a \,\overline{*}^{[0]}\, b = a, \quad a \,\overline{*}^{[1]}\, b = a \,\overline{*}\, b, \quad a \,\overline{*}^{[i+j]}\, b = (a \,\overline{*}^{[i]}\, b) \,\overline{*}^{[j]}\, (b \,\overline{*}^{[i]}\, b) \quad (2)$$

for $i, j \in \mathbb{Z}$. Then $\underline{*}^{[n]}$ and $\overline{*}^{[n]}$ are well-defined.

$$\left(\begin{array}{l} a \,\underline{*}^{[2]}\, b = (a \,\underline{*}\, b) \,\underline{*}\, (b \,\underline{*}\, b), a \,\underline{*}^{[3]}\, b = ((a \,\underline{*}\, b) \,\underline{*}\, (b \,\underline{*}\, b)) \,\underline{*}\, ((b \,\underline{*}\, b) \,\underline{*}\, (b \,\underline{*}\, b)), \\ a \,\underline{*}^{[-1]}\, b = a \,\underline{*}^{-1}\, (b \,\underline{*}^{[-1]}\, b), a \,\underline{*}^{[-2]}\, b = (a \,\underline{*}^{[-1]}\, b) \,\underline{*}^{[-1]}\, (b \,\underline{*}^{[-1]}\, b, \end{array} \right).$$

A biquandle X is of type m if

$$m = \min\{l > 0 \mid a \,\underline{*}^{[l]}\, b = a = a \,\overline{*}^{[l]}\, b \ (\forall a, b \in X)\}.$$

If $\nexists n$, then we call it of type ∞.

Proposition 6. *For a biquandle $(X, \underline{*}, \overline{*})$ of type m, $(X, (\underline{*}^{[n]})_{n \in \mathbb{Z}_m}, (\overline{*}^{[n]})_{n \in \mathbb{Z}_m})$ is a $\mathbb{Z}_m$-family of biquandles.*

Proposition 7. *R: a ring, G: a group with identity e, $\varphi : G \to Z(G)$ a homomorphism, where $Z(G)$ is the center of G. X: a right $R[G]$-module. Then X is a G-family of biquandles with $x \underline{*}^g y = xg + y(\varphi(g) - g)$, $x \overline{*}^g y = x\varphi(g)$, which we call a G-family of Alexander biquandles.*

X: a G-family of biquandles. $G_x := \{x\} \times G$. Then $\coprod_{x \in X} G_x$ is a partially multiplicative biquandle with

$$(x, g) \underline{*} (y, h) = (x \underline{*}^h y, h^{-1}gh), (x, g) \overline{*} (y, h) = (x \overline{*}^h y, g)$$

for $\forall x, y \in X$, $\forall g, h \in G_x$.