The space of short ropes and the classifying space of the space of long knots

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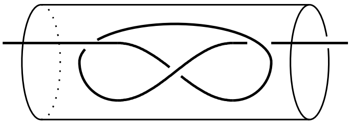
The space of long knots

D^2 : the unit *open* disk

Long knots : embeddings $f : \mathbb{R}^1 \hookrightarrow \mathbb{R}^1 \times D^2$ (or their images) satisfying

$$x \notin [0,1] \implies f(t) = (t, 0, 0).$$



$\mathcal{K}$:= {long knots} with C^∞-topology

Fact. $\pi_0(\mathcal{K}) = \mathcal{K}/\text{isotopy} \cong \{\text{knots in } S^3\}/\text{isotopy}$.

$\mathcal{K}$ is a topological monoid via concatenation (connected-sum). Thus

- $\pi_0(\mathcal{K})$ is a monoid, and
- the **classifying space** $B\mathcal{K}$ can be defined (later).

Main Theorem. $B\mathcal{K}$ is weakly equivalent to the space $\mathcal{R}$ of **short ropes**.

Corollary (J. Mostovoy, 2002). $\pi_1(\mathcal{R}) \cong \widehat{\pi_0(\mathcal{K})}$ (the group completion).

Classification of loops on $\mathcal{R}$ up to homotopy	$\longleftrightarrow$	classification of knots up to isotopy

The space of short ropes

Ropes : embeddings $r : [0,1] \hookrightarrow \mathbb{R}^1 \times D^2$ satisfying $r(0) = (0,0,0)$ and $r(1) = (1,0,0)$.

a short rope

Short ropes : ropes of length < 3.

a non-short rope

$\mathcal{R} :=$ {short ropes} with C^∞-topology

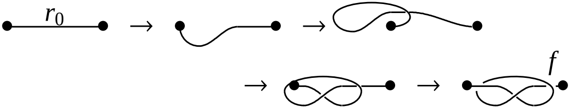
$\pi_0(\mathcal{R}) = \{r_0\}$, $r_0(t) := (t,0,0)$ (the tight rope)

no classification problem of ropes

r_0

Generators of $\pi_1(\mathcal{R}) \cong \widehat{\pi_0(\mathcal{K})}$ (Mostovoy). For $f \in \mathcal{K}$,

(1) tie f around $(0,0,0)$;



(2) unknot f around $(1,0,0)$ in a “reversed way”

Classifying spaces of (topological) categories

For a (topological) category $\mathscr{C}$,

$N_k\mathscr{C} := \{(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} c_k)$; composable k morphisms$\} \subset \mathrm{Mor}_{\mathscr{C}}^{\times k}$

$N_*\mathscr{C} := \{N_k\mathscr{C}\}_{k\geq 0}$ (the nerve of $\mathscr{C}$) is a *simplicial space* via compositions / insertion of identities.

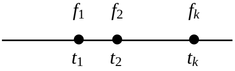
Definition. $B\mathscr{C} := |N_*\mathscr{C}| = \bigsqcup_{k\geq 0}\left(N_k\mathscr{C} \times \Delta^k\right)/\sim$: the classifying space of $\mathscr{C}$

$\Delta^k = \{0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$

- $((f_i)_{i=1}^k; (t_i)_{i=1}^k) \in N_k\mathscr{C} \times \Delta^k$ gives an element of $B\mathscr{C}$
- $t_i = t_{i+1} \implies ((f_i)_i; (t_i)_i) = (\ldots, f_{i+1} \circ f_i, \ldots ; \ldots, t_i, t_{i+2}, \ldots) \in B\mathscr{C}$
- $f_i = \mathrm{id} \implies ((f_i)_i; (t_i)_i) = (\ldots, f_{i-1}, f_{i+1}, \ldots ; \ldots, t_i, t_{i+2}, \ldots) \in B\mathscr{C}$

In the following

- $\mathscr{C} = \mathcal{K}$: the category of long knots,
- $f_i \iff$ long knots,
- composition $\iff$ connected-sum



“Connected-sum of long knots”

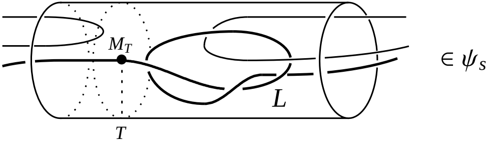
The space ψ_s of "long" 1-manifolds

$M_A := M \cap (A \times D^2)$ for $A \subset \mathbb{R}^1$ and a manifold $M \subset \mathbb{R}^1 \times D^2$

Definition (S. Galatius, O. Randal-Williams, 2010).

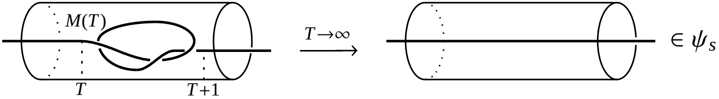
$\psi_s := \{M^1 \subset \mathbb{R}^1 \times D^2$ w/o boundary |

- M_T is compact for $\forall T \in \mathbb{R}^1$,
- $\forall$ connected component of M is "long" in at least one direction of $\mathbb{R}^1$,
- exactly one comp. $L \subset M$ is "long" in both directions; $L_T \neq \emptyset$ for $\forall T \in \mathbb{R}^1$,
- $\exists$ at least one $T \in \mathbb{R}^1$ s.t. M_T is a one point set $\}$



Topologize ψ_s so that "M is close to N if they are close in a compact set."

Example.



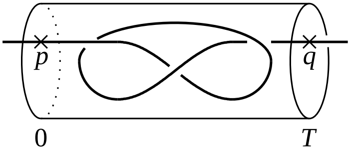
The space of long knots as a topological category

The category $\mathcal{K}$;

$\mathrm{Ob}(\mathcal{K}) = D^2$,

$\mathrm{Mor}_{\mathcal{K}}(p, q) = \{(T, M) \in \mathbb{R}^1_{\geq 0} \times \psi_s \mid M$ connected,

$\exists \epsilon > 0$ s.t. $M_{(-\infty, \epsilon]} = \{p\} \times (-\infty, \epsilon], \quad M_{[T-\epsilon, \infty)} = \{q\} \times [T - \epsilon, \infty)\}$



- $\mathrm{Mor}_{\mathcal{K}}(p, q) \simeq \{$long knots$\}$,
- $N_k\mathcal{K} = \{(0 \leq T_1 \leq \cdots \leq T_k; f) \mid f_{T_i}$ are one point sets$\}$.

  $(f = f_{[0,T_1]} \# f_{[T_1,T_2]} \# \cdots \# f_{[T_{k-1},T_k]})$

Want to know $B\mathcal{K} = |N_*\mathcal{K}|$.

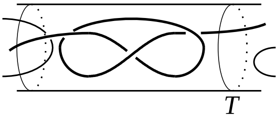
The space of long knots as a topological category

The partially ordered sets (posets) $\mathcal{D}$ and $\mathcal{D}^{\perp}$;

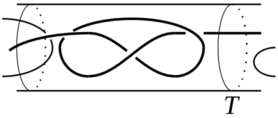
$\mathcal{D} := \{(T, M) \in \mathbb{R} \times \psi_s \mid M_T$ is a one point set$\}$,

$\mathcal{D}^{\perp} := \{(T, M) \in \mathcal{D} \mid \exists \epsilon > 0$ s.t. $M_{(T-\epsilon, T+\epsilon)} = M_T \times (T - \epsilon, T + \epsilon)\} \subset \mathcal{D}$,

$(T, M) \leq (T', M') \overset{\text{def}}{\iff} M = M'$ and $T \leq T'$



$\in \mathcal{D}$



$\in \mathcal{D}^{\perp}$

Posets are categories;

$$\mathrm{Ob}(\mathcal{D}^{(\perp)}) := \mathcal{D}^{(\perp)}, \quad \mathrm{Mor}_{\mathcal{D}^{(\perp)}}(x, y) := \begin{cases} \{*\} & x \leq y \in \mathcal{D}^{(\perp)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

$\mathrm{Mor}_{\mathcal{D}} = \{(T_0 \leq T_1; M) \mid M_{T_i}$ are one point sets$\}$,

$N_k\mathcal{D} = \{(T_0 \leq \cdots \leq T_k; M) \mid M_{T_i}$ are one point sets$\}$

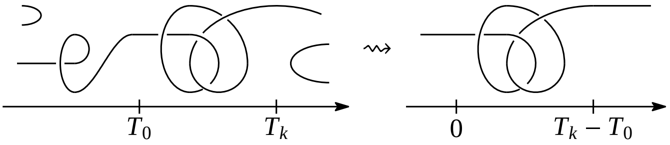
Remark. All the “half-long” components $\subset ((-\infty, T_0] \sqcup [T_k, \infty)) \times D^2$.

The classifying space of long knots

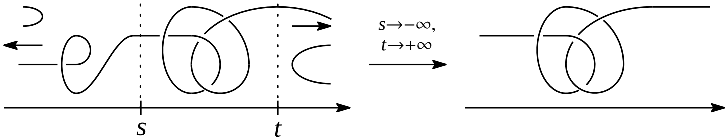
Theorem (essentially due to S. Galatius and O. Randal-Williams).

$\exists$ simplicial maps $N_*\mathcal{D} \xleftarrow{\simeq} N_*\mathcal{D}^\perp \xrightarrow{\simeq} N_*\mathcal{K}$ that are levelwise homotopy equivalences. Thus $B\mathcal{D} \xleftarrow{\simeq} B\mathcal{D}^\perp \xrightarrow{\simeq} B\mathcal{K}$.

Main point: $N_k\mathcal{D}^\perp \to N_k\mathcal{K} \to N_k\mathcal{D}^\perp$ is given by "cut-off"



This is homotopic to id by the definition of the topology of ψ_s;



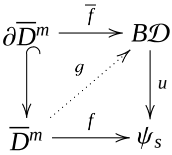
ψ_s is the classifying space of long knots

$\exists u : B\mathcal{D} \to \psi_s$, induced by $N_*\mathcal{D} \times \Delta^* \ni ((T_i)_i; M), (t_i)_i) \mapsto M$.

Theorem (essentially due to Galatius and Randal-Williams). The map u is a weak equivalence. Thus $B\mathcal{K} \sim \psi_s$.

Outline of proof. Want to show $\pi_m(\psi_s, B\mathcal{D}) = 0$ for $\forall m$.

Given the strict arrows $\overset{?}{\Longrightarrow}$ $\exists$ the dotted g ?



$\forall a \in \mathbb{R}$, $U_a := \{x \in \overline{D}^m \mid f(x)_a$ is a one point set$\}$

$\Longrightarrow$ $\{U_a\}_{a \in \mathbb{R}}$ is an open covering of $\overline{D}^m$.

Pick a finitely many subcover $\mathcal{U} = \{U_{a_i}\}_i$ and a partition of unity $\{\lambda_i\}_i$ subordinate to $\mathcal{U}$.

Roughly $g(x) := (((a_i)_i; f(x)), (\lambda_i(x))_i) \in N_*\mathcal{D} \times \Delta^*$.

ψ_s is the space of short ropes

$\mathcal{R} :=$ {short ropes} $\ni r \implies$ length$(r) < 3$

Remark. $\mathcal{R} \hookrightarrow \{r$: rope $\mid r_t$ is a one point set for $\exists t \in (0,1)\}$.

a short rope a non-short rope

Lemma. The above inclusion is a weak equivalence.

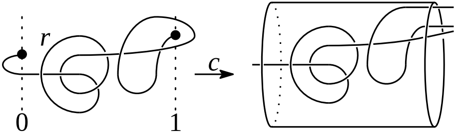
Below $\mathcal{R} := \{r$: rope $\mid r_t$ is a one point set for $\exists t \in (0,1)\}$.

Fix $f : (0,1) \xrightarrow{\approx} \mathbb{R}$.

Theorem (Moriya-S). The "cut-off" map $c : \mathcal{R} \to \psi_s$,

$$c(r) := (f \times \mathrm{id}_{D^2})(r_{(0,1)}),$$

is a weak equivalence. Thus $B\mathcal{K} \sim \mathcal{R}$.



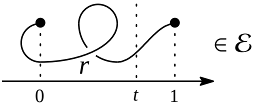
ψ_s is the space of short ropes

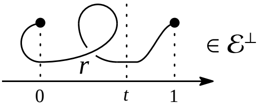
The posets (categories) $\mathcal{E}^\perp \subset \mathcal{E}$;

$\mathcal{E} := \{(t, r) \in (0, 1) \times \mathcal{R} \mid r_t$ is a one point set$\}$

$\mathcal{E}^\perp := \{(t, r) \in \mathcal{E} \mid \exists\epsilon > 0$ s.t. $r_{(t-\epsilon,t+\epsilon)} = r_t \times (t - \epsilon, t + \epsilon)\}$

$(t, r) \le (t', r') \overset{\text{def}}{\iff} r = r',\ t \le t'$





$\mathrm{Mor}_{\mathcal{E}^{(\perp)}} = \{(0 < t_0 \le t_1 < 1; r) \mid r_{t_i}$ are one point sets$\}$

$N_k\mathcal{E} = \{(0 < t_0 \le \cdots \le t_k < 1; r) \mid r_{t_i}$ are one point sets$\}$

Theorem (essentially due to Galatius and Randal-Williams).

$\exists$(weak) equivalences $B\mathcal{E}^\perp \xrightarrow{\simeq} B\mathcal{E} \xrightarrow{\sim} \mathcal{R}$.

Proof. Similar to the proof of $B\mathcal{D}^\perp \xrightarrow{\simeq} B\mathcal{D} \xrightarrow{\sim} \psi_s$.

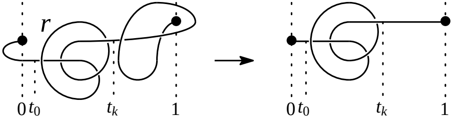
ψ_s is the space of short ropes

Theorem. A simplicial map $\Phi : N_*\mathcal{E}^\perp \to N_*\mathcal{D}^\perp$,

$\Phi(t_0 \leq \cdots \leq t_k; r) := (T_0 \leq \cdots \leq T_k; c(r))$ where $T_i := f(t_i)$

is a levelwise homotopy equivalence. Thus $B\mathcal{E}^\perp \xrightarrow{\simeq} B\mathcal{D}^\perp$.

Main point: $N_k\mathcal{E}^\perp \xrightarrow{\Phi} N_k\mathcal{D}^\perp \to N_k\mathcal{E}^\perp$ "unknots r around the endpoints"



This is homotopic to id; $\exists$ a canonical way to unknot $r_{(-\infty,t_0]}$ and $r_{[t_k,\infty)}$ (Mostovoy).

Conclusion.

$$\begin{array}{ccccc} \mathcal{R} & \xrightarrow[\sim]{c} & \psi_s & & \\ \uparrow \sim & \circlearrowleft & \uparrow \sim & & \\ B\mathcal{E}^\perp & \xrightarrow[\simeq]{\Phi} & B\mathcal{D}^\perp & \xrightarrow[\text{"cut-off"}]{\sim} & B\mathcal{K} \end{array}$$