Polynomial Invariants of Quasi-Alternating Links

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Outline

1. Quasi-alternating links
2. The Q-polynomial of QA links
3. The Jones polynomial of QA links
The set $\mathcal{Q}$ of quasi-alternating links is the smallest set such that:

1. The unknot belongs to $\mathcal{Q}$.
2. If $L$ is a link with a diagram $D$ having a crossing $c$ such that
   - Both smoothing of $D$ at $c$, $L_0$ and $L_\infty$ are in $\mathcal{Q}$,
   - $\det(L_0), \det(L_\infty) \geq 1$,
   - $\det(L) = \det(L_0) + \det(L_\infty)$; then $L \in \mathcal{Q}$

We say that $L$ is quasi-alternating at the crossing $c$ with quasi-alternating diagram $D$. 

\begin{align*}
  L & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Examples

1) Any Alternating non-split link is QA (at any crossing in any alternating diagram)
2) The knot $9_{47}$ is a QA non-alternating knot. Here is a QA diagram of $9_{47}$, at the indicated crossing.
Properties

1. The branched double-cover of a QA link is an $L$-space [Ozsváth and Szabó];
2. The space of branched double-cover of a QA link bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$, [Ozsváth and Szabó];
3. The $\mathbb{Z}/2\mathbb{Z}$ knot Floer homology group of a QA link is thin [Manolescu and Ozsváth];
4. The reduced ordinary Khovanov homology group of a QA link is thin [Manolescu and Ozsváth];
5. The reduced odd Khovanov homology group of a QA link is thin, [Ozsváth, Rasmussen and Szabó];
The Jones polynomial

The Jones polynomial is an isotopy invariant of oriented links defined by:

\[
V_{\bigcirc}(t) = 1
\]

\[
tV_{L_+}(t) - t^{-1}V_{L_-}(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)V_{L_0}(t),
\]

where \(L_+, L_-\) and \(L_0\) are 3 links as pictured below:

We define the \textit{span}(V_L) as the difference between the highest and the lowest degree of \(t\) that appear in \(V_L(t)\).
Motivation: Alternating links

- If $L$ is an alternating link, then $\text{span}(V_L) = c(L)$, where $c(L)$ is the crossing number of $L$.
- $V_L(t)$ is an alternating polynomial.
- The coefficients of the highest and lowest degree in $V_L(t)$ are both $\pm 1$
- The Jones polynomial of a quasi-alternating link is also alternating. This can be seen as a consequence of its thin Khovanov homology.
- How about $\text{span}V_L$ if $L$ is a quasi-alternating link?
For any link $L$, $Q_L(x)$ is a Laurent polynomial which can be defined by $Q_{\bigcirc}(x) = 1$ and a recursive relation on link diagrams as follows:

$$Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_\infty}(x))$$

where $L_+$, $L_-$, $L_0$ and $L_\infty$ are four links which are identical except in a small ball where they are as in the following picture:
Q-polynomials

- $Q_L(x) = F_L(1, x)$ where $F$ is the two variable Kauffman polynomial.
- The constant term in $Q_L(x)$ is odd. Consequently $\deg Q_L \geq 0$.
- If $U$ is the unlink with $k$ components then $Q_U(x) = (2x^{-1} - 1)^{k-1}$. 
Theorem

**Theorem [Qazaqzeh-C: AGT 2015]**

For any quasi-alternating link $L$, we have $\deg Q_L \leq \det(L) - 1$, where $\det(L)$ is the determinant of $L$.

**Example.** For the knot $10_{132}$ we have:

$$Q_L(x) = 5 - 18x - 14x^2 + 38x^3 + 20x^4 - 24x^5 - 12x^6 + 4x^7 + 2x^8.$$  

$$\det(L) = \sqrt{Q_L(2)} = 5$$  

$\deg(Q_L) = 8 > \det(L) = 5$, then $10_{132}$ is not Quasi-alternating.
**Remark.** Our theorem does not characterize quasi-alternating links since the knot $10_{128}$ for instance satisfies the inequality $\deg(Q_L) < \det(L)$, but not quasi-alternating since it is homologically thick in Khovanov homology. (same for the knots $9_{46}, 11_{n50}$)
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<td>$9_{42}$</td>
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<td>$11n135$</td>
<td>5</td>
<td>7</td>
<td>$11n139$</td>
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### Quasi-alternating links

#### The Q-polynomial of QA links

<table>
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<tr>
<th>Link</th>
<th>Det.</th>
<th>Deg.</th>
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<td>6</td>
</tr>
<tr>
<td>$L_{8n8}$</td>
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<td>5</td>
</tr>
<tr>
<td>$L_{9n9}$</td>
<td>4</td>
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</tr>
<tr>
<td>$L_{9n15}$</td>
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<td>7</td>
</tr>
<tr>
<td>$L_{9n19}$</td>
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<td>5</td>
</tr>
<tr>
<td>$L_{9n27}$</td>
<td>4</td>
<td>7</td>
</tr>
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</table>

#### The Jones polynomial of QA links

<table>
<thead>
<tr>
<th>Link</th>
<th>Det.</th>
<th>Deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{8n6}$</td>
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<td>6</td>
</tr>
<tr>
<td>$L_{9n4}$</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>$L_{9n12}$</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$L_{9n18}$</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$L_{9n21}$</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>
Let $K(p, q)$ be the Kanenobu knot. For any $(p, q)$ we have $\det(K(p, q)) = 25$. The degree of the $Q$-polynomial of any Kanenobu knot is computed by [Qazaqzeh and Mansour]:

$$\deg Q(K(p, q)) = \begin{cases} 
|p| + |q| + 6, & \text{if } pq \geq 0, \\
|p| + |q| + 5, & \text{otherwise}, 
\end{cases}$$

**Corollary.** There are only finitely many Kanenobu knots which are quasi alternating.

*Green’s conjecture !!!*
Improvement and Extension by Teragaito

Theorem [Teragaito]

Let \( L \) be a Quasi-alternating link which is not a \((2,n)\)-torus links. Then:
\[
\deg Q_L < \det(L) - 1.
\]

**Example:** For knot \( K = 10_{140} \), we have \( \det(K) = 9 \) and:
\[
Q_K(x) = 2x^8 + 4x^7 - 12x^6 - 22x^5 + 24x^4 + 32x^3 - 24x^2 - 12x + 9.
\]
So by Teragaito’s Theorem \( K \) is not quasi-alternating. However, the non quasi-alternateness of \( K \) is not detected by the condition given in Qazaqzeh-C.

Similar results have been obtained by Teragaito for the two-variable Kauffman polynomial of QA links (other than \((2,n)\)) torus:
\[
\text{Deg}_z(F_L) \leq \det(L) - 2.
\]
Sketch of the Proof of the main Theorem

We first prove the following Lemma:

**Lemma**

Let $L$ be a link, then $\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1$, where $L_0, L_\infty$ are the smoothings of the link $L$ at any crossing $c$.

Then we use induction on the determinant of $L$. At a QA crossing, we have:

\[
\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1 \\
< \max\{\det(L_0), \det L_\infty\} + 1 \\
< \det(L_0) + \det(L_\infty) = \det(L).
\]
The Jones polynomial is an isotopy invariant of oriented links defined by:

\[ V\bigcirc(t) = 1 \]
\[ tV_{L_+}(t) - t^{-1}V_{L_-}(t) = (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}(t), \]

where \( L_+, L_- \) and \( L_0 \) are 3 links as pictured below:

We define the \( \text{span}(V_L) \) as the difference between the highest and the lowest degree of \( t \) that appear in \( V_L(t) \).
Conjecture

If $L$ is a quasi-alternating link, then $\text{Span}V_L \leq \det(L)$.

- This conjecture is weaker than the QQJ Conjecture: If $L$ is QA then $c(L) \leq \det(L)$, where $c(L)$ is the crossing number of $L$.
- The conjecture is True for any alternating (non split) link.
- The conjecture is True for any quasi-alternating link with braid index less than or equal to 3.
Proof of the Conjecture for closed 3-braids

$B_n$ be the braid group on $n$ strings with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the following relations:

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2
\]
\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \forall \ 1 \leq i \leq n - 2.
\]

Figure: The generators $\sigma_1$ and $\sigma_2$ of $B_3$ respectively

Polynomial Invariants of Quasi-Alternating Links

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Closed 3-braids have been classified by Murasugi

**Theorem [Murasugi]**

Let $b$ be a 3-braid and let $h = (\sigma_1 \sigma_2)^3$ be a full positive twist. Then $b$ is conjugate to exactly one of the following:

1. $h^n \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s}$, where $s$, $p_i$ and $q_i$ are positive integers.
2. $h^n \sigma_2^m$ where $m \in \mathbb{Z}$.
3. $h^n \sigma_1^m \sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$. 
Theorem [Baldwin]

Let $L$ be a closed 3-braid, then

1. If $L$ is the closure of $h^n \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_s} \sigma_2^{-q_s}$, where $s$, $p_i$ and $q_i$ are positive integers, then $L$ is quasi-alternating if and only if $n \in \{-1, 0, 1\}$.

2. If $L$ is the closure of $h^n \sigma_2^m$, then $L$ is quasi-alternating if and only if either $n = 1$ and $m \in \{-1, -2, -3\}$ or $n = -1$ and $m \in \{1, 2, 3\}$.

3. If $L$ is the closure of $h^n \sigma_1^m \sigma_2^{-1}$ where $m \in \{-1, -2, -3\}$. Then $L$ is quasi-alternating if and only if $n \in \{0, 1\}$. 
Proposition. Suppose that \( L = h^n \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s} \), where \( s, p_i \) and \( q_i \) are positive integers. Let \( p = \sum_{i=1}^{s} p_i \) and \( q = \sum_{i=1}^{s} q_i \).

If \( n \) is odd, then
\[
\det(L) = 4 + pq + \sum_{k=2,i_1<\cdots<i_k} p_{i_1} \cdots p_{i_k} (q_{i_1} + \cdots + q_{i_2-1}) \cdots (q_{i_{k-1}} + \cdots + q_{i_k-1}) \cdot q - (q_{i_1} + \cdots + q_{i_k-1}) .
\]

If \( n \) is even, then
\[
\det(L) = pq + \sum_{k=2,i_1<\cdots<i_k} p_{i_1} \cdots p_{i_k} (q_{i_1} + \cdots + q_{i_2-1}) \cdots (q_{i_{k-1}} + \cdots + q_{i_k-1}) \cdot q - (q_{i_1} + \cdots + q_{i_k-1}) .
\]
Proof of Proposition

- Use Tait graph to compute the determinant of the alternating link $L' = \sigma_1^{p_1} \sigma_2^{-q_1} \ldots \sigma_1^{p_s} \sigma_2^{-q_s}$

- Then use Birman’s formula for the Jones polynomial of closed 3-braids to compute the determinant of the link $L$.

**Remark.** For each of the two other cases in Baldwin’s theorem, we have explicit formula for the Jones polynomial, hence for the determinant.
Proof for 3-braids

Case by case check of the $SpanV_L$ (or the crossing number) and the determinant.