Polynomial Invariants of Quasi-Alternating Links

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Outline

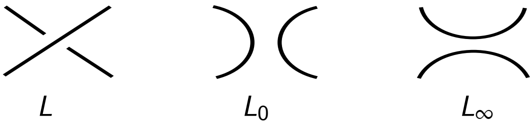
QA links

Definition

The set $\mathcal{Q}$ of quasi-alternating links is the smallest set such that:

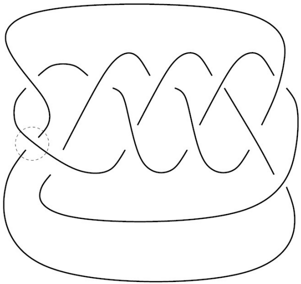
- The unknot belongs to $\mathcal{Q}$.
- If L is a link with a diagram D having a crossing c such that
  1. Both smoothing of D at c, L_0 and L_∞ are in $\mathcal{Q}$,
  2. $\det(L_0), \det(L_\infty) \geq 1$,
  3. $\det(L) = \det(L_0) + \det(L_\infty)$; then $L \in \mathcal{Q}$

  We say that L is quasi-alternating at the crossing c with quasi-alternating diagram D.



Examples

1) Any Alternating non-split link is QA (at any crossing in any alternating diagram)
2) The knot 9_{47} is a QA non-alternating knot. Here is a QA diagram of 9_{47}, at the indicated crossing.



Properties

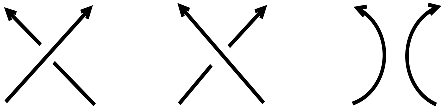
1. The branched double-cover of a QA link is an L-space [Ozsváth and Szabó];
2. The space of branched double-cover of a QA link bounds a negative definite 4-manifold W with $H_1(W) = 0$, [Ozsváth and Szabó];
3. The $\mathbb{Z}/2\mathbb{Z}$ knot Floer homology group of a QA link is thin [Manolescu and Ozsváth];
4. The reduced ordinary Khovanov homology group of a QA link is thin [Manolescu and Ozsváth];
5. The reduced odd Khovanov homology group of a QA link is thin,[Ozsváth, Rasmussen and Szabó];

The Jones polynomial

The Jones polynomial is an isotopy invariant of oriented links defined by:

$$V_{\bigcirc}(t) = 1$$
$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) = (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}(t),$$

where L_+, L_- and L_0 are 3 links as pictured below:



We define the *span*(V_L) as the difference between the highest and the lowest degree of t that appear in $V_L(t)$.

Motivation: Alternating links

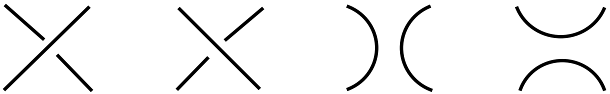
- If L is an alternating link, then $span(V_L) = c(L)$, where $c(L)$ is the crossing number of L.
- $V_L(t)$ is an alternating polynomial.
- The coefficients of the highest and lowest degree in $V_L(t)$ are both ∓ 1
- The Jones polynomial of a quasi-alternating link is also alternating. This can be seen as a consequence of its thin Khovanov homology.
- How about $spanV_L$ if L is a quasi-alternating link?

Q-polynomials

For any link L, $Q_L(x)$ is a Laurent polynomial which can be defined by $Q_{\bigcirc}(x) = 1$ and a recursive relation on link diagrams as follows:

$$Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_\infty}(x))$$

where L_+, L_-, L_0 and L_∞ are four links which are identical except in a small ball where they are as in the following picture



Q-polynomials

- $Q_L(x) = F_L(1, x)$ where F is the two variable Kauffman polynomial.
- The constant term in $Q_L(x)$ is odd. Consequently $degQ_L \geq 0$.
- If U is the unlink with k components then $Q_U(x) = (2x^{-1} - 1)^{k-1}$.

Theorem

Theorem [Qazaqzeh-C: AGT 2015]

For any quasi-alternating link L, we have $\deg Q_L \leq \det(L) - 1$, where $\det(L)$ is the determinant of L.

Example. For the knot 10_{132} we have:

$$Q_L(x) = 5 - 18x - 14x^2 + 38x^3 + 20x^4 - 24x^5 - 12x^6 + 4x^7 + 2x^8.$$

$det(L) = \sqrt{Q_L(2)} = 5$

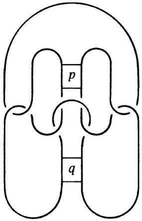
$\deg(Q_L) = 8 > det(L) = 5$, then 10_{132} is not Quasi-alternating.

Remark. Our theorem does not characterize quasi-alternating links since the knot 10_{128} for instance satisfies the inequality $\deg(Q_L) < det(L)$, but not quasi-alternating since it is homologically thick in Khovanov homology.
(same for the knots 9_{46}, 11_{n50})

Knot	Det.	Deg.	Knot	Det.	Deg.
8_{19}	3	6	9_{42}	7	7
10_{124}	1	8	10_{132}	5	8
10_{139}	3	8	10_{145}	3	8
10_{153}	1	8	10_{161}	5	6
$11n9$	5	9	$11n19$	5	9
$11n31$	3	9	$11n34$	1	9
$11n38$	3	9	$11n42$	1	9
$11n49$	1	9	$11n57$	7	9
$11n67$	9	9	$11n96$	7	9
$11n102$	3	9	$11n104$	3	9
$11n111$	7	9	$11n116$	1	7
$11n135$	5	7	$11n139$	9	9

Link	Det.	Deg.	Link	Det.	Deg.
$L8n3$	4	6	$L8n6$	0	6
$L8n8$	0	5	$L9n4$	4	7
$L9n9$	4	7	$L9n12$	6	7
$L9n15$	2	7	$L9n18$	2	7
$L9n19$	0	5	$L9n21$	4	6
$L9n27$	4	7			

Application to Kanenobu Knots



Let $K(p,q)$ be the Kanenobu knot
For any (p,q) we have $\det(K(p,q)) = 25$.
The degree of the Q-polynomial of any Kanenobu knot is computed by [Qazaqzeh and Mansour]:

$$\deg Q(K(p,q)) = \begin{cases} |p| + |q| + 6, & \text{if } pq \geq 0, \\ |p| + |q| + 5, & \text{otherwise,} \end{cases}$$

Corollary. There are only finitely many Kanenobu knots which are quasi alternating.

Green's conjecture !!!

Improvement and Extension by Teragaito

Theorem [Teragaito]

Let L be a Quasi-alternating link which is not a $(2, n)-$torus links. Then:
$degQ_L < det(L) - 1$.

Example: For knot $K = 10_{140}$, we have $\det(K) = 9$ and :
$Q_K(x) =$
$2x^8 + 4x^7 - 12x^6 - 22x^5 + 24x^4 + 32x^3 - 24x^2 - 12x + 9$. So by Teragaito's Theorem K is not quasi-alternating. However, The non quasi-alternateness of K is not detected by the condition given in Qazaqzeh-C.
Similar results have been obtained by Teragaito for the two-variable Kauffman polynomial of QA links (other than $(2, n)$) torus : $Deg_z(F_L) \leq det(L) - 2$.

Sketch of the Proof of the main Theorem

We first prove the following Lemma:

Lemma

Let L be a link, then $\deg Q_L \leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1$, where L_0, L_∞ are the smoothings of the link L at any crossing c.

Then we use induction on the determinant of L.
At a QA crossing, we have:

$$\begin{aligned} \deg Q_L &\leq \max\{\deg Q_{L_0}, \deg Q_{L_\infty}\} + 1 \\ &< \max\{\det(L_0), \det L_\infty\} + 1 \\ &< \det(L_0) + \det(L_\infty) = \det(L). \end{aligned}$$

The Jones polynomial is an isotopy invariant of oriented links defined by:

$$V_{\bigcirc}(t) = 1$$
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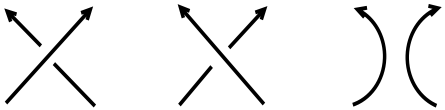
where L_+, L_- and L_0 are 3 links as pictured below:



We define the $span(V_L)$ as the difference between the highest and the lowest degree of t that appear in $V_L(t)$.

Conjecture

If L is a quasi-alternating link, then $SpanV_L \leq \det(L)$.

- This conjecture is weaker than the QQJ Conjecture: If L is QA then $c(L) \leq \det(L)$, where $c(L)$ is the crossing number of L.
- The conjecture is True for any alternating (non split) link.
- The conjecture is True for any quasi-alternating link with braid index less than or equal to 3.

Proof of the Conjecture for closed 3-braids

B_n be the braid group on n strings with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \forall\ 1 \leq i \leq n-2.$$

Figure: The generators σ_1 and σ_2 of B_3 respectively

Closed 3-braids have been classified by Murasugi

Theorem [Murasugi]

Let b be a 3-braid and let $h = (\sigma_1\sigma_2)^3$ be a full positive twist. Then b is conjugate to exactly one of the following:

1. $h^n\sigma_1^{p_1}\sigma_2^{-q_1}\ldots\sigma_1^{p_s}\sigma_2^{-q_s}$, where s, p_i and q_i are positive integers.
2. $h^n\sigma_2^m$ where $m \in \mathbb{Z}$.
3. $h^n\sigma_1^m\sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$.

Theorem [Baldwin]

Let L be a closed 3-braid, then

1. If L is the closure of $h^n\sigma_1^{p_1}\sigma_2^{-q_1}\ldots\sigma_1^{p_s}\sigma_2^{-q_s}$, where s, p_i and q_i are positive integers, then L is quasi-alternating if and only if $n \in \{-1, 0, 1\}$.
2. If L is the closure of $h^n\sigma_2^m$, then L is quasi-alternating if and only if either $n = 1$ and $m \in \{-1, -2, -3\}$ or $n = -1$ and $m \in \{1, 2, 3\}$.
3. If L is the closure of $h^n\sigma_1^m\sigma_2^{-1}$ where $m \in \{-1, -2, -3\}$. Then L is quasi-alternating if and only if $n \in \{0, 1\}$.

Proposition. Suppose that $L = h^n\widehat{\sigma_1^{p_1}\sigma_2^{-q_1}\ldots\sigma_1^{p_s}\sigma_2^{-q_s}}$, where s, p_i and q_i are positive integers. Let $p = \sum_{i=1}^{s} p_i$ and $q = \sum_{i=1}^{s} q_i$.

If n is odd, then

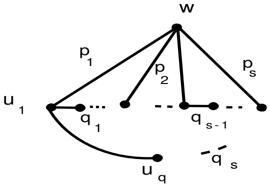
$$det(L) = 4 + pq + \sum_{k=2, i_1<\cdots<i_k}^{s} p_{i_1}\ldots p_{i_k}(q_{i_1}+\cdots+q_{i_2-1})\ldots(q_{i_{k-1}}+\cdots+q_{i_k-1})(q-(q_{i_1}+\cdots+q_{i_{k-1}})).$$

If n is even, then

$$det(L) = pq + \sum_{k=2,, i_1<\cdots<i_k}^{s} p_{i_1}\ldots p_{i_k}(q_{i_1}+\cdots+q_{i_2-1})\ldots(q_{i_{k-1}}+\cdots+q_{i_k-1})(q-(q_{i_1}+\cdots+q_{i_{k-1}}))$$

Proof of Proposition

- Use Tait graph to compute the determinant of the alternating link $L' = \widehat{\sigma_1^{p_1}\sigma_2^{-q_1}\ldots\sigma_1^{p_s}\sigma_2^{-q_s}}$



- Then use Birman's formula for the Jones polynomial of closed 3-braids to compute the determinant of the link L.

Remark. For each of the two other cases in Baldwin's theorem, we have explicit formula for the Jones polynomial, hence for the determinant.

Proof for 3-braids

Case by case check of the $Span V_L$ (or the crossing number)and the determinant.