強可逆結び目の不変サイフェルト曲面

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We first recall basic definitions in knot theory.

**Definition (Seifert Surface)**

A **Seifert surface** for a knot $K \subset S^3$ is an embedded orientable surface $S \subset S^3$ with $\partial S = K$.

**Theorem (Seifert’s Theorem)**

Every knot in $S^3$ bounds a Seifert surface.

**Definition (Genus)**

The **genus** of a knot $K$ is defined:

$$g(K) := \min \{ g(S) \mid S : \text{a Seifert surface for } K \}.$$
A knot $K \subset S^3$ is called a **periodic knot of period** $n$ if there exists a periodic map $\varphi : (S^3, K) \to (S^3, K)$ of period $n$ such that

- $\text{Fix}(\varphi) \cong S^1$,
- $\text{Fix}(\varphi) \cap K = \emptyset$. 
Strongly Invertible Knot

Definition (Strongly Invertible Knot)

A knot $K$ is **strongly invertible** if there exists an inversion $h : (S^3, K) \rightarrow (S^3, K)$ such that

- $\text{Fix}(h) = S^1$,
- $\text{Fix}(h) \cap K = \{2 \text{ pts}\}$.

![Diagram of a strongly invertible knot](image_url)
Theorem [Edmonds-Livingston, 1983]

For any periodic knot $K \subset S^3$ with a periodic map $\varphi$, there exists an “incompressible” Seifert surface $S$ for $K$ such that $\varphi(S) = S$. In particular, if $K$ is a fibered knot, then $S$ is a minimal genus Seifert surface.

So it is natural to ask the following question.

Question

What about for strongly invertible knots?
Fact

There is a strongly invertible knot which admits no invariant Seifert surface of minimal genus.

This knot has exactly two minimal genus Seifert surfaces $S_1$ and $S_2$ up to isotopy.

But the strong inversion $h$ interchanges $S_1$ and $S_2$. 
Does every strongly invertible knot \((K, h)\) have an invariant Seifert surface? Here an invariant Seifert surface for \((K, h)\) is a Seifert surface for \(K\) such that \(h(S) = S\).

**Remark:** If \(S\) is an invariant Seifert surface for \(K\), then \(S \cap \text{Fix}(h)\) is a sub-arc of \(\text{Fix}(h) \cong S^1\) bounded by \(\text{Fix}(h) \cap K = S^0\).
For a strongly invertible knot \((K, h)\), let \(\delta_1\) and \(\delta_2\) be the sub-arcs of \(\text{Fix}(h)\) bounded by \(\text{Fix}(h) \cap K\).

For each \(i = 1, 2\), does there exist an invariant Seifert surface \(S_i\) for \((K, h)\) such that \(S_i \cap \text{Fix}(h) = \delta_i\)?

Result 1 (H)

Yes.

In first part of this talk, we give a positive answer to this question. There is an algorithm to construct an invariant Seifert surface for a given strongly invertible knot.
Can the gaps between the invariant genera and the genera be arbitrarily large?

Yes.

\[ \exists \{K_n\}_{n \in \mathbb{N}}; \forall N \in \mathbb{N}, \exists n \in \mathbb{N}; g(K_n, h, \delta_i) - g(K_n) > N. \]

The invariant genus of \((K, h, \delta_i)\) is defined:

\[ g(K, h, \delta_i) := \min \{ g(S) \mid S: \text{an } h\text{-invariant Seifert surface with } \delta_i \subset S \}. \]
Basic Observation (1/2)

\((K, h)\): a strongly invertible knot.

\(\pi : S^3 \to S^3 / h \cong S^3\).

\(O := \pi(\text{Fix}(h)), \quad \delta'_i := \pi(\delta_i), \quad k := \pi(K)\).

\(\tilde{S}\): an invariant Seifert surface for \((K, h)\) containing \(\delta_1 \subset \text{Fix}(h)\).

Then \(S := \pi(\tilde{S})\) is a (possibly non-orientable) surface in \(S^3 / h\) satisfying the following two conditions.

**Condition (i)** \(\partial S = \delta'_1 \cup k, \quad S \cap O = \partial S \cap O = \delta'_1\).
Basic Observation (2/2)

**Condition (ii)** \( \forall \gamma \subset \text{int}(S'): \) a loop,
- \( \gamma \) is an orientation preserving loop \( \iff \text{lk}(\gamma, O) \equiv 0 \, (\text{mod} \ 2) \),
- \( \gamma \) is an orientation reversing loop \( \iff \text{lk}(\gamma, O) \equiv 1 \, (\text{mod} \ 2) \).

**Proposition**

If \( S \subset S^3/h \) is a surface satisfying Conditions (i) and (ii), then \( \tilde{S} := \pi^{-1}(S) \) is an invariant Seifert surface for \((K, h)\).
An Algorithm to Construct An Invariant Seifert Surface

\((K, h)\): a strongly invertible knot.
\(\pi: S^3 \rightarrow S^3/h \cong S^3\).
\(O := \pi(\text{Fix}(h)), \delta'_i := \pi(\delta_i), k := \pi(K)\).
\(\theta(K, h) := k \cup O\).

An Algorithm (1/3)

**Step 1.** Modify \(\theta(K, h)\) as in the following figure.

---

\(\text{Fix}(h)\)

\(\delta_1\)

\(\delta_2\)

\(\delta'_1\)

\(\delta'_2\)

---

\(k\)

\(O\)

\(\pi\)
An Algorithm to Construct An Invariant Seifert Surface

An Algorithm (2/3)

Step 2. Modify $\theta(K, h)$ as in the following figure around “straps” for $\theta(K, h)$.

Step 3. Modify further $\theta(K, h)$ to make the number of “straps” even.

Step 4. Fix an orientation of $k$, and number the “straps” according to the orientation. Rearrange the “straps” by isotopy, so that they link $\delta'_2 \subset O$ from the top to the bottom according to the order.
Step 5. Attach the bands \( \{B_i\} \) for \( \delta'_1 \) and each pair of two successive “straps.”

Step 6. By cutting off the bands \( \{B_i\} \) constructed in Step 5, we obtain a split link \( O \cup \tilde{k} \) from \( \theta(K, h) \). By applying Seifert’s algorithm to \( \tilde{k} \), we obtain the Seifert surface \( \tilde{S} \) for \( \tilde{k} \) which is separated from \( O \).

Then \( S := \tilde{S} \cup (\bigcup B_i) \) satisfies Conditions (i) and (ii). Hence \( \tilde{S} := \pi^{-1}(S) \) is an invariant Seifert surface for \( (K, h) \).
An Example

\[ K \xrightarrow{\pi} K' \]

\[ \tilde{S} \xrightarrow{\pi} S \]
**The Conway Notation**

$K$: a 2-bridge knot.

If we describe $K$ as in the following figure, then $K$ is denoted by $C(a_1, \ldots, a_n)$.

\[a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \cdots \rightarrow a_n\]

$n$: odd

\[a_n \rightarrow a_n\]

$n$: even

-3 = \[\text{figure} \]

2 = \[\text{figure} \]
Invariant Seifert Surface for 2-Bridge Knots

\[ K = C(2b_1, \ldots, 2b_{2k}). \]

\( h \): a strong inversion as in figure.

Then

**Theorem (H)**

\[ g(K, h, \delta_1) = \sum_{i: \text{odd}} |b_i|. \]

In particular,

\[ g(K, h, \delta_1) - g(K) = \sum_{i: \text{odd}} |b_i| - k = \sum_{i: \text{odd}} (|b_i| - 1). \]
\( \tilde{S} \): an invariant Seifert surface for \((K, h)\) with \( \delta_1 \subset \tilde{S} \).

\[ S := \pi(\tilde{S}). \]

\( N(\delta'_1) \subset S \): a regular neighborhood of \( \delta'_1 \).

\[ S' := \text{cl}(S - N(\delta'_1)), \quad k' := \partial S', \quad K' := k' \cup O. \]

Note that \( k' \) might “link” \( O \) around \( \delta'_1 \).

Observe that \( K' := k' \cup O \) is the 2-bridge link
\[ C(4b_1, b_2, 4b_3, b_4, \ldots, b_{2k}, 2m) \] for some \( m \in \mathbb{Z} \).
Normalization

\[ S_t^2 := S^2 \times \{ t \} \subset S^2 \times \mathbb{R} \subset S^3. \]

We assume that \( K' = O \cup k' \subset S^2 \times [0, 1] \subset S^3 \) satisfies the following conditions:

- \( K' \cap S_1^2 \) is a pair of mutually disjoint arcs of slope 1/0.
- \( K' \cap S_0^2 \) is a pair of mutually disjoint arcs of slope \( p/q \).
- \( K' \cap S_t^2 (\forall t \in (0, 1)) \) consists of four points.
- \( \#(O \cap S_t^2) = 2, \#(k' \cap S_t^2) = 2 \).
\(\tilde{S}\): an invariant Seifert surface for \((K, h)\).

\[ S := \pi(\tilde{S}). \]

\(N(\delta'_1) \subset S\): a regular neighborhood of \(\delta'_1\).

\[ S' := \text{cl}(S - N(\delta'_1)), \quad k' := \partial S, \quad K' := k' \cup O. \]

\textbf{Claim 1}

\[\chi(\tilde{S}) = 2\chi(S) - 1 \quad \text{and} \quad \chi(S) = \chi(S').\]

\[
\begin{align*}
S &= S' \cup I, \\
\tilde{S} &= \tilde{S}' \cup \tilde{B}.
\end{align*}
\]

\[
\chi(\tilde{S}) = \chi(\tilde{S}') + \chi(\tilde{B}) - 2\chi(I) = 2\chi(S') + 1 - 2 = 2\chi(S) - 1.
\]
\[
\tilde{S} \subset S^3: \text{ an invariant Seifert surface such that } g(\tilde{S}) = g(K, h, \delta_1).
\]

Then

**Claim 2**

\[ S' \subset S^3 \setminus N(K') \text{ is incompressible and } \partial\text{-incompressible surface} \]

satisfying Conditions (i)’ and (ii)’.

**Condition (i)’** \[ \partial S' = k', S' \cap O = \emptyset. \]

**Condition (ii)’** \[ \forall \gamma \subset \text{int}(S'): \text{ a loop,} \]

\[ \begin{align*}
\gamma \text{ is an orientation preserving loop } & \iff \text{lk}(\gamma, O) \equiv 0 \text{ (mod 2)}, \\
\gamma \text{ is an orientation reversing loop } & \iff \text{lk}(\gamma, O) \equiv 1 \text{ (mod 2)}. \end{align*} \]
\[ S' := \text{cl}(S - N(\delta'_1)). \]
We regard a saddle of \( S' \) in the way as a band \( B = I \times I \) attached at its two ends \( (\partial I) \times I \) to \( S' \).

**Claim 3 (cf. Hatcher-Thurston, 1985)**

For each \( t \in (0, 1) \), \( S' \cap S^2_t \) is an arc \( \alpha \) such that \( \partial \alpha \subset k' \) and \( \alpha \cap O = \emptyset \).

**Claim 4**

Each saddle of \( S' \) has the following form up to homeo.
\(\lambda_0 = 1/0, \lambda_1, \ldots, \lambda_l = p/q: \) the sequence of slopes of \(S' \cap S_t^2\) from the top to the bottom s.t. \(\lambda_i \neq \lambda_{i+1}\).

Then

Claim 5 (cf. Hatcher-Thurston, 1985)

If \(S'\) is incompressible and \(\partial\)-incompressible, then it can be isotoped (rel \(K'\)) such that \(\lambda_i \neq \lambda_{i+2}\) for each \(i\).

Since the dual graph of the Farey tessellation is a tree, we can evaluate the number of saddles.

Hence we can calculate the minimum genus of an invariant Seifert surface for 2-bridge knots.
The Farey tessellation is the diagram as in the following figure. There is an edge joining two fractions \( a/b \) and \( c/d \) whenever \( ad - bc = \pm 1 \). The edge from \( a/b \) to \( c/d \) is the long side of triangle whose third vertex is \((a + c)/(b + d)\). There is the sequence of triangles from \( 1/0 \) to \( p/q \).
Evaluation of Saddles

\[ K = C(2b_1, \ldots, 2b_{2k}), \quad K' = C(4b_1, b_2, 4b_3, b_4, \ldots, b_{2k}, 2m). \]

The relative condition between the slopes of \( \alpha_{t-\varepsilon} \) and \( \alpha_{t+\varepsilon} \) is as in the following figures.

Since the dual graph of the Farey tessellation is a tree, we need at least
\[ \sum_{i: \text{odd}} |b_i| \] saddles.
Calculation of the Invariant Genus

\[ K = C'(2b_1, \ldots, 2b_{2k}). \]
\[ n := \sum_{i: \text{odd}} |b_i|. \]

Thus \( S' \) obtains from \( D^2 \) by attaching \( n \) bands.

Here,

\[ \chi(S') = \chi(D^2) + n\chi(\text{band}) - 2n\chi(I) \]
\[ = 1 + n - 2n \]
\[ = 1 - n. \]

By using Claim 1,

\[ \chi(\widetilde{S}) = 2\chi(S') - 1 \]
\[ = 2(1 - n) - 1 \]
\[ = 2 - 2n. \]

Hence,

\[ g(\widetilde{S}) = n = \sum_{i: \text{odd}} |b_i|. \]
Result 2

\[ K = C(2b_1, \ldots, 2b_{2k}). \]

**Theorem (H)**

\[
g(K, h, \delta_1) = \sum_{i: \text{odd}} |b_i|.
\]

In particular,

\[
g(K, h, \delta_1) - g(K) = \sum_{i: \text{odd}} |b_i| - k = \sum_{i: \text{odd}} (|b_i| - 1).
\]
**Question**

Can $g(K, h, \delta_2) - g(K)$ and $g(K, h) - g(K)$ be also arbitrarily large?

**Definition (Invariant Genus)**

The invariant genus of $(K, h)$ is defined:

$$g(K, h) := \min_{i=1, 2} \{g(K, h, \delta_i)\}.$$

$$K_n := C(4, 4, \ldots, 4)_{2n}$$

Then

$$g(K_n, h) - g(K_n) = n?$$
Thank You for Your Attention!