強可逆結び目の不変ザイフェルト曲面

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Basic Definitions

We first recall basic definitions in knot theory.

Definition (Seifert Surface)

A **Seifert surface** for a knot $K \subset S^3$ is an embedded orientable surface $S \subset S^3$ with $\partial S = K$.

Theorem (Seifert's Theorem)

Every knot in S^3 bounds a Seifert surface.

Definition (Genus)

The **genus** of a knot K is defined:

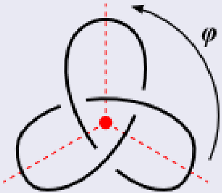
$$g(K) := \min\{g(S) \mid S: \text{ a Seifert surface for } K\}.$$

Periodic Knot

Definition (Periodic Knot)

A knot $K \subset S^3$ is called a **periodic knot of period n** if there exists a periodic map $\varphi : (S^3, K) \to (S^3, K)$ of period n such that

- $\mathrm{Fix}(\varphi) \cong S^1$,
- $\mathrm{Fix}(\varphi) \cap K = \emptyset$.

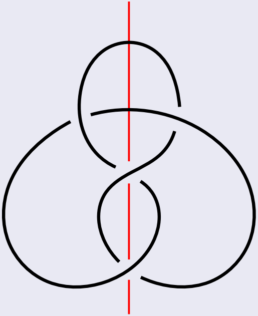


Strongly Invertible Knot

Definition (Strongly Invertible Knot)

A knot K is **strongly invertible** if there exists an inversion $h : (S^3, K) \to (S^3, K)$ such that

- $\mathrm{Fix}(h) = S^1$,
- $\mathrm{Fix}(h) \cap K = \{2 \text{ pts}\}$.



Previous Study

Theorem [Edmonds-Livingston, 1983]

For any periodic knot $K \subset S^3$ with a periodic map φ, there exists an "incompressible" Seifert surface S for K such that $\varphi(S) = S$.
In particular, if K is a fibered knot, then S is a minimal genus Seifert surface.

So it is natural to ask the following question.

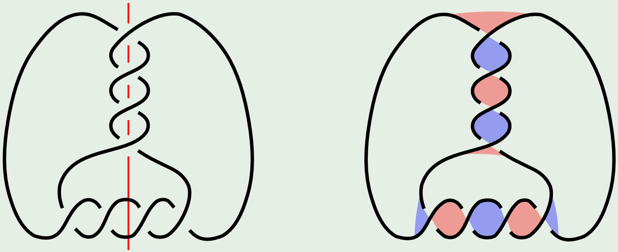
Question

What about for strongly invertible knots?

Fact

Fact

There is a strongly invertible knot which admits no invariant Seifert surface of minimal genus.

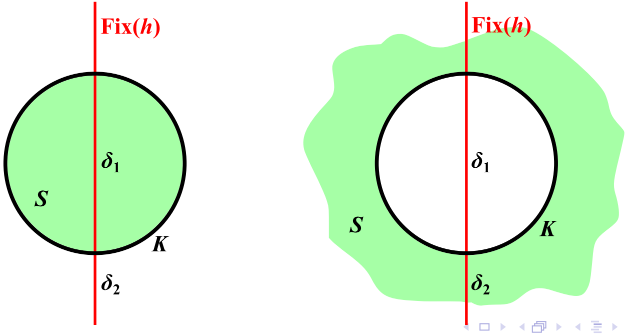


This knot has exactly two minimal genus Seifert surfaces S_1 and S_2 up to isotopy.
But the strong inversion h interchanges S_1 and S_2.

Question 1

Does every strongly invertible knot (K, h) have an invariant Seifert surface? Here an invariant Seifert surface for (K, h) is a Seifert surface for K such that $h(S) = S$.

Remark: If S is an invariant Seifert surface for K, then $S \cap \mathrm{Fix}(h)$ is a sub-arc of $\mathrm{Fix}(h) \cong S^1$ bounded by $\mathrm{Fix}(h) \cap K = S^0$.



Question 1 (Refined)

Question 1 (Refined)

For a strongly invertible knot $(K,\ h)$, let δ_1 and δ_2 be the sub-arcs of $\mathrm{Fix}(h)$ bounded by $\mathrm{Fix}(h) \cap K$.
For each $i = 1,\ 2$, does there exist an invariant Seifert surface S_i for $(K,\ h)$ such that $S_i \cap \mathrm{Fix}(h) = \delta_i$?

Result 1 (H)

Yes.

In first part of this talk, we give a positive answer to this question.
There is an algorithm to construct an invariant Seifert surface for a given strongly invertible knot.

Question 2

Can the gaps between the invariant genera and the genera be arbitrarily large?

Result 2 (H)

Yes.

$$\exists \{K_n\}_{n\in\mathbb{N}};\ \forall N \in \mathbb{N},\ \exists n \in \mathbb{N};\ g(K_n,\ h,\ \delta_i) - g(K_n) > N.$$

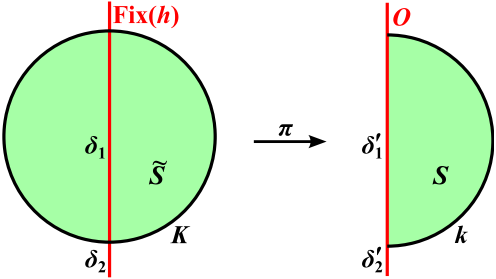
Definition (Invariant Genus)

The invariant genus of $(K,\ h,\ \delta_i)$ is defined:

$$g(K,\ h,\ \delta_i) := \min\{g(S) \mid S\text{: an } h\text{-invariant Seifert surface with } \delta_i \subset S\}.$$

(K, h): a strongly invertible knot.
$\pi : S^3 \to S^3/h \cong S^3$.
$O := \pi(\mathrm{Fix}(h)),\ \delta_i' := \pi(\delta_i),\ k := \pi(K)$.
$\widetilde{S}$: an invariant Seifert surface for (K, h) containing $\delta_1 \subset \mathrm{Fix}(h)$.
Then $S := \pi(\widetilde{S})$ is a (possibly non-orientable) surface in S^3/h satisfying the following two conditions.

Condition (i) $\partial S = \delta_1' \cup k,\ S \cap O = \partial S \cap O = \delta_1'$.



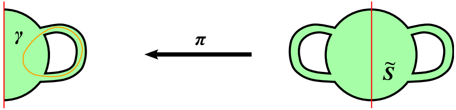
Basic Observation (2/2)

Condition (ii) $\forall \gamma \subset \mathrm{int}(S)$: a loop,

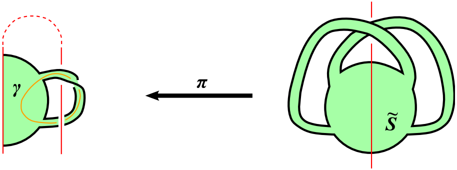
γ is an orientation preserving loop $\iff \mathrm{lk}(\gamma, O) \equiv 0 \pmod 2$,

γ is an orientation reversing loop $\iff \mathrm{lk}(\gamma, O) \equiv 1 \pmod 2$.

lk(γ, O) is even:



lk(γ, O) is odd:



Proposition

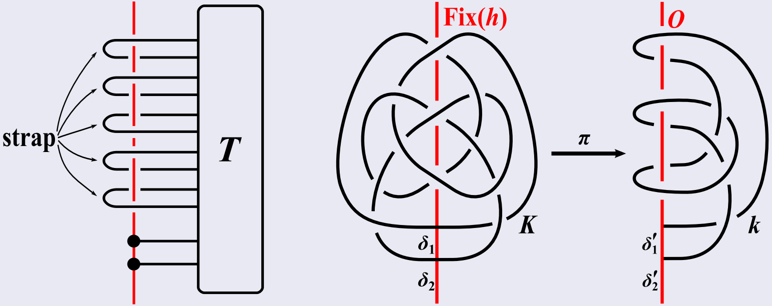
If $S \subset S^3/h$ is a surface satisfying Conditions (i) and (ii), then $\widetilde{S} := \pi^{-1}(S)$ is an invariant Seifert surface for $(K,\, h)$.

An Algorithm to Construct An Invariant Seifert Surface

(K, h): a strongly invertible knot.
$\pi : S^3 \to S^3/h \cong S^3$.
$O := \pi(\mathrm{Fix}(h)),\ \delta_i' := \pi(\delta_i),\ k := \pi(K)$.
$\theta(K, h) := k \cup O$.

An Algorithm (1/3)

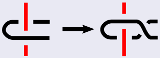
Step 1. Modify $\theta(K, h)$ as in the following figure.

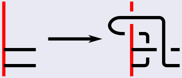


An Algorithm to Construct An Invariant Seifert Surface

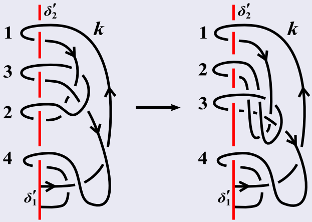
An Algorithm (2/3)

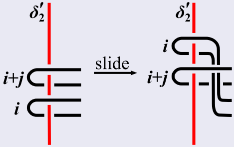
Step 2. Modify $\theta(K,\,h)$ as in the following figure around "straps" for $\theta(K,\,h)$.
Step 3. Modify further $\theta(K,\,h)$ to make the number of "straps" even.





Step 4. Fix an orientation of k, and number the "straps" according to the orientation. Rearrange the "straps" by isotopy, so that they link $\delta_2' \subset O$ from the top to the bottom according to the order.



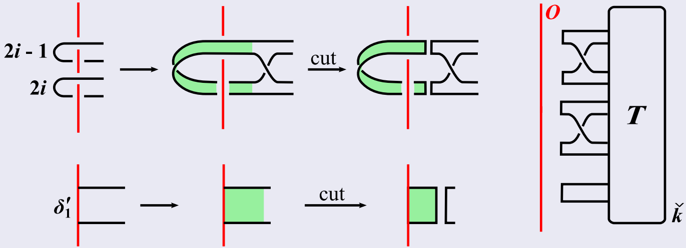


An Algorithm (3/3)

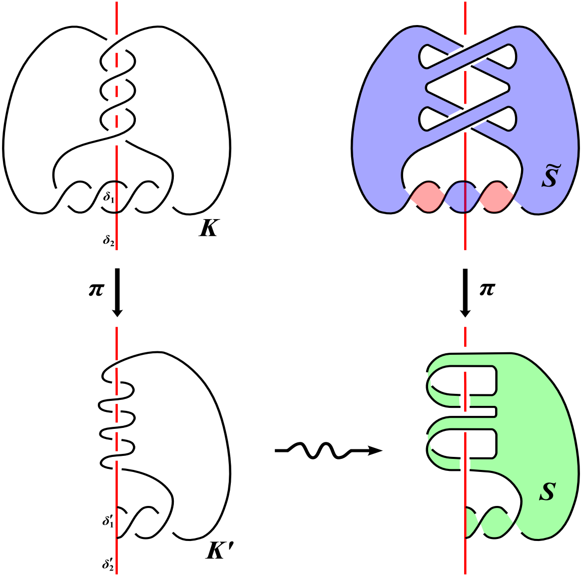
Step 5. Attach the bands $\{B_i\}$ for δ_1' and each pair of two successive "straps."

Step 6. By cutting off the bands $\{B_i\}$ constructed in *Step 5*, we obtain a split link $O \cup \check{k}$ from $\theta(K, h)$. By applying Seifert's algorithm to $\check{k}$, we obtain the Seifert surface $\check{S}$ for $\check{k}$ which is separated from O.

Then $S := \check{S} \cup (\bigcup B_i)$ satisfies Conditions (i) and (ii). Hence $\widetilde{S} := \pi^{-1}(S)$ is an invariant Seifert surface for (K, h).

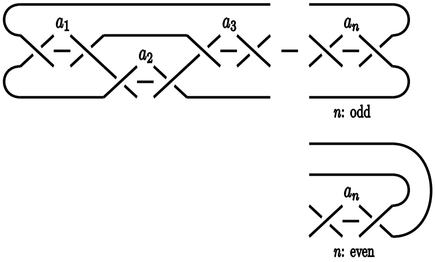


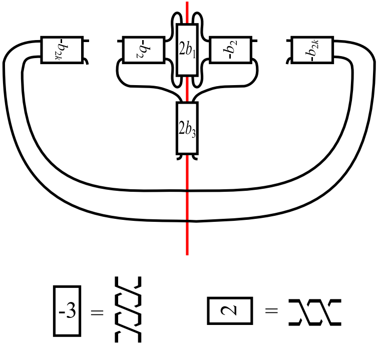
An Example



The Conway Notation

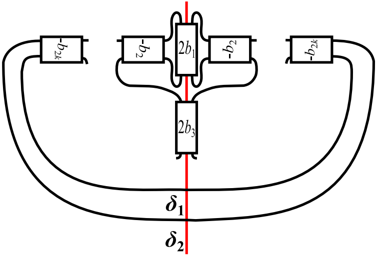
K: a 2-bridge knot.
If we describe K as in the following figure, then K is denoted by $C(a_1, \ldots, a_n)$.





Invariant Seifert Surface for 2-Bridge Knots

$K = C(2b_1, \ldots, 2b_{2k})$.
h: a strong inversion as in figure.
Then



Theorem (H)

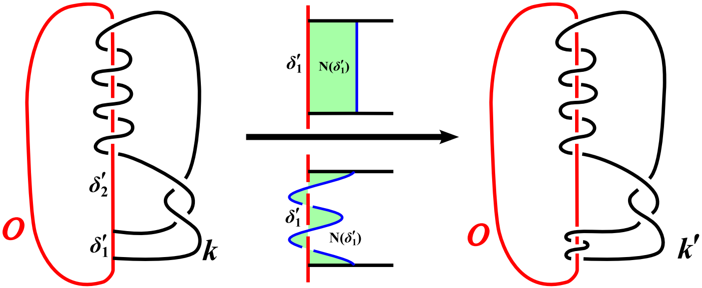
$$g(K,\, h,\, \delta_1) = \sum_{i:\ odd} |b_i|.$$

In particular,

$$g(K,\, h,\, \delta_1) - g(K) = \sum_{i:\ odd} |b_i| - k = \sum_{i:\ odd} (|b_i| - 1).$$

Basic Observation

$\widetilde{S}$: an invariant Seifert surface for (K, h) with $\delta_1 \subset \widetilde{S}$.
$S := \pi(\widetilde{S})$.
$\mathrm{N}(\delta_1') \subset S$: a regular neighborhood of δ_1'.
$S' := \mathrm{cl}(S - \mathrm{N}(\delta_1')),\ k' := \partial S',\ K' := k' \cup O$.
Note that k' might "link" O around δ_1'.



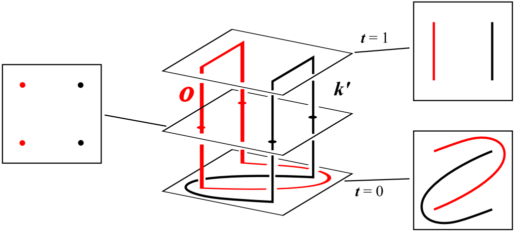
Observe that $K' := k' \cup O$ is the 2-bridge link
$C(4b_1, b_2, 4b_3, b_4, \ldots, b_{2k}, 2m)$ for some $m \in \mathbb{Z}$.

Normalization

$S^2_t := S^2 \times \{t\} \subset S^2 \times \mathbb{R} \subset S^3$.

We assume that $K' = O \cup k' \subset S^2 \times [0,\,1] \subset S^3$ satisfies the following conditions:

- $K' \cap S^2_1$ is a pair of mutually disjoint arcs of slope $1/0$.
- $K' \cap S^2_0$ is a pair of mutually disjoint arcs of slope p/q.
- $K' \cap S^2_t$ $(\forall t \in (0,\,1))$ consists of four points.
- $\#(O \cap S^2_t) = 2,\ \#(k' \cap S^2_t) = 2$.



Claims (1/4)

$\widetilde{S}$: an invariant Seifert surface for $(K,\, h)$.
$S := \pi(\widetilde{S})$.
$\mathrm{N}(\delta_1') \subset S$: a regular neighborhood of δ_1'.
$S' := \mathrm{cl}(S - \mathrm{N}(\delta_1')),\ k' := \partial S,\ K' := k' \cup O$.

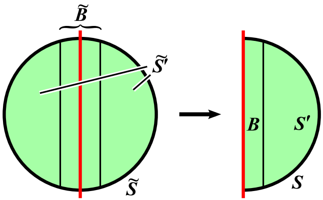
Claim 1

$\chi(\widetilde{S}) = 2\chi(S) - 1$ and $\chi(S) = \chi(S')$.

$$S = S' \underset{I}{\cup} B.$$

$$\widetilde{S} = \widetilde{S}' \underset{2I}{\cup} \widetilde{B}.$$

$$\begin{aligned}\chi(\widetilde{S}) &= \chi(\widetilde{S}') + \chi(\widetilde{B}) - 2\chi(I) \\ &= 2\chi(S') + 1 - 2 \\ &= 2\chi(S) - 1.\end{aligned}$$



$\widetilde{S} \subset S^3$: an invariant Seifert surface such that $g(\widetilde{S}) = g(K,\, h,\, \delta_1)$.
Then

Claim 2

$S' \subset S^3/h - \mathrm{N}(K')$ is incompressible and ∂-incompressible surface satisfying Conditions (i)' and (ii)'.

Condition (i)' $\quad \partial S' = k',\ S' \cap O = \emptyset$.

Condition (ii)' $\quad \forall \gamma \subset \mathrm{int}(S')$: a loop,

γ is an orientation preserving loop $\iff \mathrm{lk}(\gamma,\, O) \equiv 0 \pmod 2$,
γ is an orientation reversing loop $\iff \mathrm{lk}(\gamma,\, O) \equiv 1 \pmod 2$.

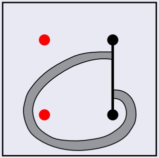
$S' := \mathrm{cl}(S - \mathrm{N}(\delta_1'))$.
We regard a saddle of S' in the way as a band $B = I \times I$ attached at its two ends $(\partial I) \times I$ to S'.

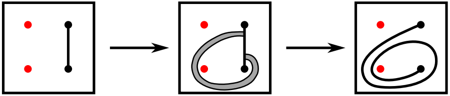
Claim 3 (cf. Hatcher-Thurston, 1985)

For each $t \in (0,\ 1)$, $S' \cap S_t^2$ is an arc α such that $\partial\alpha \subset k'$ and $\alpha \cap O = \emptyset$.

Claim 4

Each saddle of S' has the following form up to homeo.





$\lambda_0 = 1/0, \lambda_1, \ldots, \lambda_l = p/q$: the sequence of slopes of $S' \cap S_t^2$ from the top to the bottom s.t. $\lambda_i \neq \lambda_{i+1}$.
Then

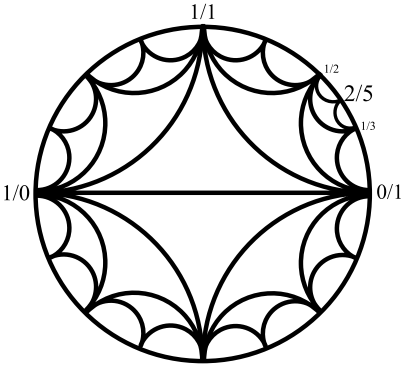
Claim 5 (cf. Hatcher-Thurston, 1985)

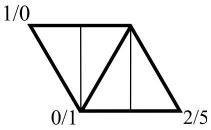
If S' is incompressible and ∂-incompressible, then it can be isotoped (rel K') such that $\lambda_i \neq \lambda_{i+2}$ for each i.

Since the dual graph of the Farey tessellation is a tree, we can evaluate the number of saddles.
Hence we can calculate the minimum genus of an invariant Seifert surface for 2-bridge knots.

Farey Tessellation

The Farey tessellation is the diagram as in the following figure.
There is an edge joining two fractions a/b and c/d whenever $ad - bc = \pm 1$.
The edge from a/b to c/d is the long side of triangle whose third vertex is $(a + c)/(b + d)$.
There is the sequence of triangles from $1/0$ to p/q.



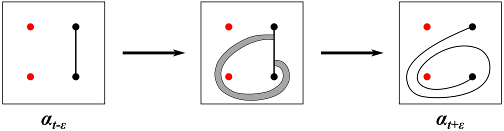


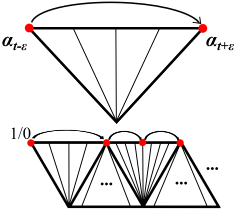
Evaluation of Saddles

$K = C(2b_1, \ldots, 2b_{2k}),\ K' = C(4b_1, b_2, 4b_3, b_4, \ldots, b_{2k}, 2m)$.

The relative condition between the slopes of $\alpha_{t-\varepsilon}$ and $\alpha_{t+\varepsilon}$ is as in the following figures.

Since the dual graph of the Farey tessellation is a tree, we need at least $\sum_{i:\ odd} |b_i|$ saddles.





Calculation of the Invariant Genus

$K = C(2b_1, \ldots, 2b_{2k})$.

$n := \sum_{i:\, odd} |b_i|$.

Thus S' obtains from D^2 by attaching n bands.

Here,

$$\begin{aligned}\chi(S') &= \chi(D^2) + n\chi(\text{band}) - 2n\chi(I)\\ &= 1 + n - 2n\\ &= 1 - n.\end{aligned}$$

By using Claim 1,

$$\begin{aligned}\chi(\widetilde{S}) &= 2\chi(S') - 1\\ &= 2(1-n) - 1\\ &= 2 - 2n.\end{aligned}$$

Hence,

$$g(\widetilde{S}) = n = \sum_{i:\, odd} |b_i|.$$

Result 2

$K = C(2b_1, \ldots, 2b_{2k})$.

Theorem (H)

$$g(K,\, h,\, \delta_1) = \sum_{i\text{: } odd} |b_i|.$$

In particular,

$$g(K,\, h,\, \delta_1) - g(K) = \sum_{i\text{: } odd} |b_i| - k = \sum_{i\text{: } odd} (|b_i| - 1).$$

Question

Can $g(K,\, h,\, \delta_2) - g(K)$ and $g(K,\, h) - g(K)$ be also arbitrarily large?

Definition (Invariant Genus)

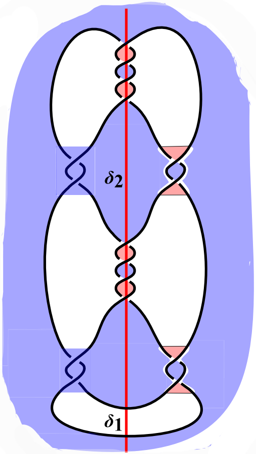
The invariant genus of $(K,\, h)$ is defined:

$$g(K,\, h) := \min_{i=1,2} \{g(K,\, h,\, \delta_i)\}.$$

$$K_n := C\,\underbrace{(4,\, 4,\, \ldots,\, 4)}_{2n}$$

Then

$$g(K_n,\, h) - g(K_n) = n?$$



Thank You for Your Attention!