Abelian subgroups of the mapping class groups for non-orientable surfaces

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Contents

Introduction

- $S = S_{g,n}$: a compact connected orientable surface of genus g with n boundary components s.t. $\chi(S) < 0$.
- $N = N_{g,n}$: a compact connected non-orientable surface of genus g with n boundary components s.t. $\chi(N) < 0$.
- $F = S$ or N.
- $\mathscr{M}(F)$: the mapping class group of F,
  i.e. the group of isotopy classes of (orientation preserving if $F = S$) self-homeomorphisms of F with isotopies fixing each boundary component of F setwise.

Introduction

Theorem (Birman-Lubotzky-McCarthy 1983)

G: any abelian subgroup of $\mathscr{M}(S)$.

Then G is finitely generated and the torsion-free rank of G is bounded by $3g+n-3$.

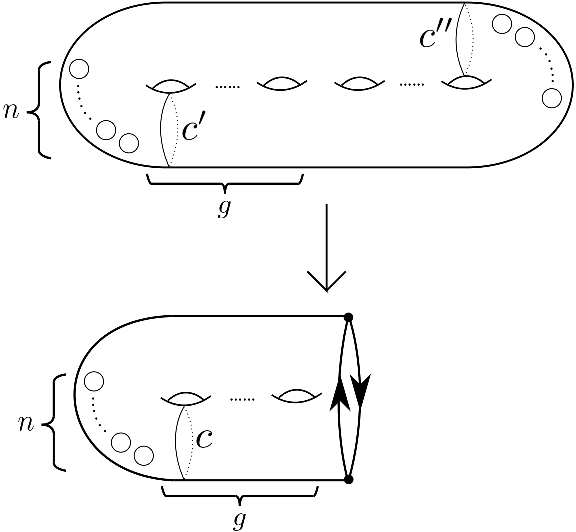
Idea of proof

To show

$\mathrm{rank}(G) \leq$ cardinality{Dehn twists along pairwise disjoint curves on S}.

Introduction

Double covering map of a non-orientable surface N.



Introduction

This double covering map induces an injective homomorphism $\iota\colon \mathscr{M}(N_{g,n}) \to \mathscr{M}(S_{g-1,2n})$.

Corollary

G: any abelian subgroup of $\mathscr{M}(N_{g,n})$.
Then G is finitely generated and the torsion-free rank of G is bounded by $3(g-1)+2n-3$.

$\longrightarrow$ This bound might not be best possible.

Introduction

Moreover

Theorem (Szepietowski 2010)

Any Dehn twists are not contained in $\iota(\mathscr{M}(N_{g,n}))$.

$\longrightarrow$ No Dehn twists in $\mathscr{M}(S_{g-1,2n})$ have lifts in $\mathscr{M}(N_{g,n})$ by ι.

$\longrightarrow$ We don't know the maximal torsion-free rank of the abelian subgroups of $\mathscr{M}(N)$ by the result of Birman-Lubotzky-McCarthy directly.

Q.

What is the maximal rank of the abelian subgroups of $\mathscr{M}(N)$?

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What is the maximal rank of the abelian subgroups of $\mathscr{M}(N)$?

Introduction

The following two theorems gave partial answers of the question.

Theorem (Atalan-Szepietowski 2014)

N: a non-orientable surface of odd genus $g \geq 5$.
Then the maximal rank of the abelian subgroups of $\mathscr{M}(N)$ is $\frac{3}{2}(g-1)+n-2$.

Theorem (Atalan 2015)

N: a non-orientable surface of even genus.
The maximal rank of the abelian subgroups of $\mathscr{M}(N)$, which contain a group generated by Dehn twists is $\frac{3}{2}g+n-3$.

$\longrightarrow$ We give the answer of this question!

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Appendix

Theorem (Ivanov 2016)

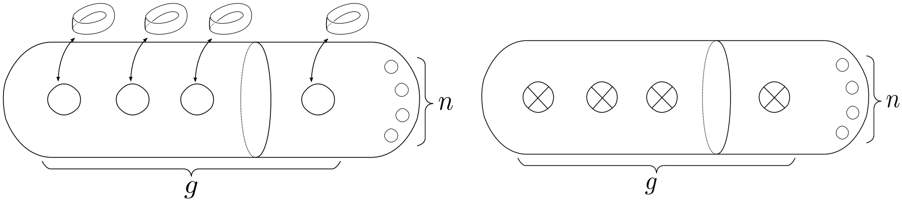
S: a closed orientable surface.

$\mathscr{I}(S)$: the Torelli group of S.

Every abelian subgroup of $\mathscr{I}(S)$ is a free abelian group with rank bounded by $2g-3$.

Preliminaries

$\diamond$ $N = N_{g,n}$: a **non-orientable surface** of genus g with n boundary components:



Preliminaries

◇ **arcs on** N:

properly embedded and essential,

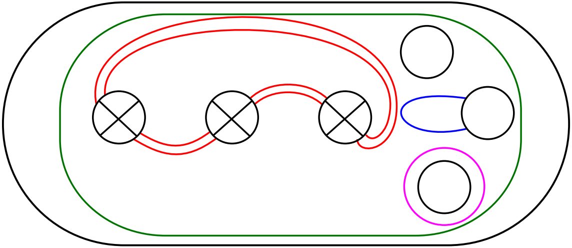
i.e. are not isotopic into ∂N

◇ **curves on** N:

properly embedded and essential,

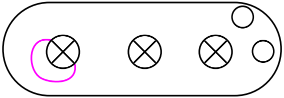
i.e. do not bound a disk or a Möbius band,

and are not isotopic to ∂N

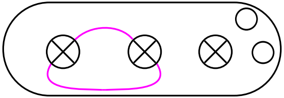


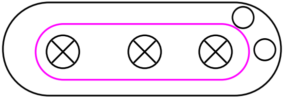
Preliminaries

◇ a one-sided curve a: the regular neighborhood of a in N is a Möbius band.



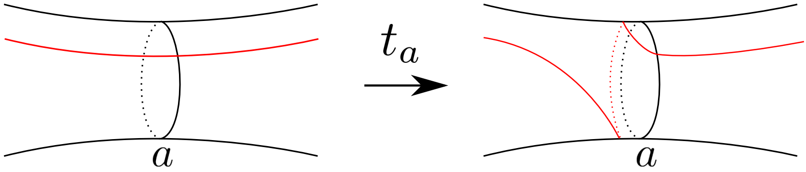
◇ a two-sided curve a: the regular neighborhood of a in N is an annulus.





Preliminaries

◇ A **Dehn twist** t_a along a two-sided curve a in N:



Preliminaries

$\diamond$ $\varphi \in \mathscr{M}(N)$ is **reducible** if $\exists f \in \varphi$ and $\exists A$: a family of curves s.t. $f(A) = A$. (We call such systems reduction systems for f.)

$\diamond$ $\varphi \in \mathscr{M}(N)$ is of **finite order** if $\exists f \in \varphi$ and $\exists n \neq 0$ s.t. $f^n = \mathrm{id}_N$.

$\diamond$ $\varphi \in \mathscr{M}(N)$ is **pseudo-Anosov** if $\exists f \in \varphi$ s.t. $\forall a$: a curve and $\forall n \neq 0$, $f^n(a) \neq a$.

Main results

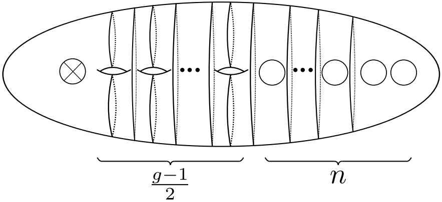
Theorem 1 (K.)

N: a non-orientable surface with $\chi(N) < 0$.

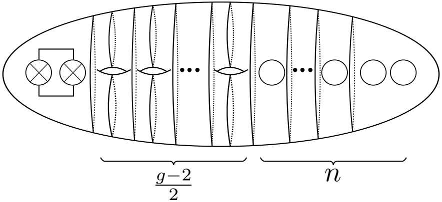
G: a torsion-free abelian subgroup of $\mathscr{M}(N)$.

Then, $G \cong \langle \tau_1, \cdots, \tau_k \rangle < \mathscr{M}(N)$, where each τ_i is an isotopy class of a Dehn twist and the supports of τ_i and τ_j are disjoint for $i \neq j$.

Further, $k \leq \frac{3}{2}(g-1) + n - 2$ if g is odd and $k \leq \frac{3}{2}g + n - 3$ if g is even.



odd genus



even genus

Differences from orientable surfaces

$\mathscr{A}$: a set of isotopy classes of curves whose representatives can be chosen to consist of pairwise disjoint.

A: a set of the representatives of $\mathscr{A}$ which are mutually disjoint.

$N_{\mathscr{A}}$: the natural compactification of $N - A$.

$\mathscr{M}_{\mathscr{A}}(N)$: the stabilizer of $\mathscr{A}$ in $\mathscr{M}(N)$.

$\longrightarrow$ We can define a well-defined homomorphism $\Lambda\colon \mathscr{M}_{\mathscr{A}}(N) \to \mathscr{M}(N_{\mathscr{A}})$ as follows:

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Differences from orientable surfaces

For $\varphi \in \mathscr{M}_{\mathscr{A}}(N)$,

we can choose a set A of the representatives of $\mathscr{A}$ and $f \in \varphi$ s.t. $f(A) = A$.
Further, $f|_{N-A}$ extends uniquely to $N_{\mathscr{A}}$ (we put it $\hat{f}$).

$\longrightarrow$ This process determines a well-defined class $\hat{\varphi} = [\hat{f}] \in \mathscr{M}(N_{\mathscr{A}})$.

Differences from orientable surfaces

$\mathscr{A}^{\mathrm{two}}$: the set of all isotopy classes of two-sided curves in $\mathscr{A}$.

$\alpha \in \mathscr{A}^{\mathrm{two}}$.

a: a representative of α.

t_a: the Dehn twist along a.

τ_α: the isotopy class of t_a.

Def.

A reduction system $\mathscr{A}$ for $\varphi \in \mathscr{M}(N)$ is adequate if each of restrictions of φ to each component of $N_{\mathscr{A}}$ is either of finite order or pseudo-Anosov.

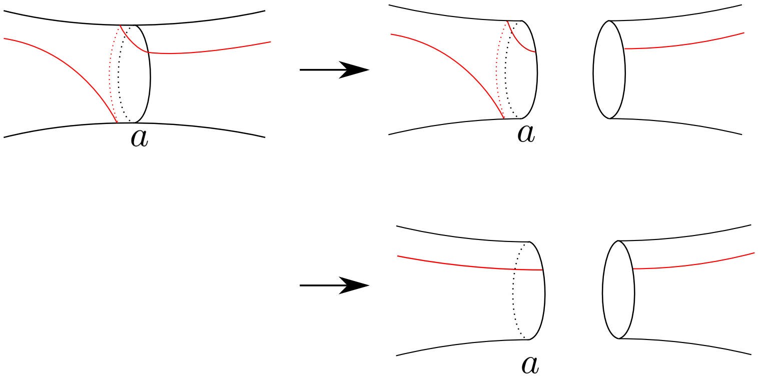
Differences from orientable surfaces

One of the differences is the following:

$\Lambda\colon \mathscr{M}_{\mathscr{A}}(N) \to \mathscr{M}(N_{\mathscr{A}})$

Lemma

$\mathrm{Ker}(\Lambda) = \langle \tau_\alpha \mid \alpha \in \mathscr{A}^{\mathrm{two}} \rangle$.



Differences from orientable surfaces

The second difference is the following:

Lemma

F: a cpt. conn. surf. with $\chi(F) < 0$.

δ: any isotopy class of properly embedded arc on F.

$\varphi \in \mathscr{M}(F)$ with $\varphi(\delta) = \delta$.

Then one of the following occurs.

(1) $F = S_{0,3}$ or $N_{1,2}$ or $N_{2,1}$.

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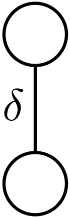
Differences from orientable surfaces

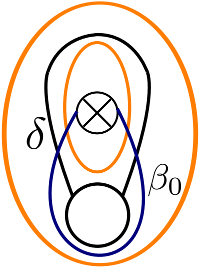
(2) If δ connects two distinct boundary components



$\longrightarrow \exists\gamma$: an isotopy class of a curve s.t. $\varphi(\gamma) = \gamma$ and $i(\alpha, \gamma) \neq 0$ for $\forall\alpha$: an isotopy class of a curve with $i(\alpha, \delta) \neq 0$.

Differences from orientable surfaces

(3) If δ is an isotopy class of an arc which connects one boundary component, goes through crosscaps even number of times, and surrounds one crosscap



$\longrightarrow$ for any α excepting β_0 with $i(\alpha, \delta) \neq 0$ there exists γ such that $\varphi(\gamma) = \gamma$ and $i(\alpha, \gamma) \neq 0$.

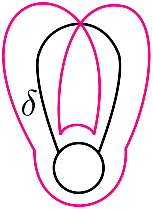
Differences from orientable surfaces

(4) If δ is an isotopy class of an arc which connects one boundary component, goes through crosscaps even number of times, and does not surround one crosscap

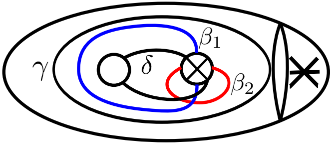
$\longrightarrow$ for any α with $i(\alpha, \delta) \neq 0$ there exists γ such that $\varphi(\gamma) = \gamma$ and $i(\alpha, \gamma) \neq 0$.

Differences from orientable surfaces

(5) If δ is an isotopy class of an arc which connects one boundary component, goes through crosscaps odd number of times



$\longrightarrow$ for any α excepting β_1 and β_2 with $i(\alpha, \delta) \neq 0$ there exists γ such that $\varphi(\gamma) = \gamma$ and $i(\alpha, \gamma) \neq 0$.



Differences from orientable surfaces

The result of the second lemma is different from that of the orientable surfaces.

However

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Differences from orientable surfaces

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Differences from orientable surfaces

Lemma

$\mathscr{A}$: an adequate reduction system for $\varphi \in \mathscr{M}(N)$.
$\alpha \in \mathscr{A}$.
$\mathscr{A}' = \mathscr{A} - \{\alpha\}$.
Then, the following are equivalent.
(1) α is an essential reduction class, where a reduction class α for φ is essential if for any isotopy class of a curve β such that $i(\alpha, \beta) \neq 0$ and $m \neq 0$, the class $\varphi^m(\beta) \neq \beta$.
(2) $\mathscr{A}'$ is not an adequate reduction system for φ^m for any $m \neq 0$.

$\longrightarrow$ We can use Birman-Lubotzky-McCarthy's techniques to the non-orientable surfaces.

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Thank you for your attention!