Minimal coloring numbers on minimal diagrams of torus links

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Let $L$ be a link, and $D$ a diagram of $L$.

\textbf{$\mathbb{Z}$-coloring}

A map $\gamma : \{\text{arcs of } D\} \rightarrow \mathbb{Z}$ is called a $\mathbb{Z}$-coloring on $D$ if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of $D$ with the over arc $a$ and the under arcs $b$ and $c$. 
**Z-colorable link**

$L$ is **Z-colorable** if $\exists$ a diagram of $L$ with a non-trivial $\mathbb{Z}$-coloring, i.e., a $\mathbb{Z}$-coloring which at least two colors.

**Remark**

$L$ is $\mathbb{Z}$-colorable if and only if $\det(L) = 0$.

Thus, any knot $K$ is non-$\mathbb{Z}$-colorable, for $\det(K)$ is odd.

**Minimal coloring number**

The **minimal coloring number** $\text{mincol}_\mathbb{Z}(D)$ of a diagram $D$ of $L$ is defined as the minimum of the number of colors among non-trivial $\mathbb{Z}$-colorings on $D$. The **minimal coloring number** $\text{mincol}_\mathbb{Z}(L)$ of $L$ is the minimum of $\text{mincol}_\mathbb{Z}(D)$. 
Let $L$ be a $\mathbb{Z}$-colorable link.

**Theorem.** [I.chihara-M., 2017]
If $L$ is non-splittable, then $\text{mincol}_{\mathbb{Z}}(L) \geq 4$.

**Proposition.**
If the crossing number of $L$ is at most 10, then $\text{mincol}_{\mathbb{Z}}(L) = 4$.

**Question:** How many colors are enough to color?
For any non-splittable $\mathbb{Z}$-colorable link $L$, $\text{mincol}_{\mathbb{Z}}(L) = 4$?
Theorem. [Zhang-Jin-Deng, 2017], [M., 2019]
For any non-splittable \( \mathbb{Z} \)-colorable link \( L \), \( \text{mincol}_{\mathbb{Z}}(L) = 4 \) holds.
Theorem. [Zhang-Jin-Deng, 2017], [M., 2019]

For any non-splittable $\mathbb{Z}$-colorable link $L$, $\text{mincol}_\mathbb{Z}(L) = 4$ holds.

(Next) Problem.

For a particular diagram $D$ of a non-splittable $\mathbb{Z}$-colorable link, how many colors are enough to color? i.e., $\text{mincol}_\mathbb{Z}(D) =$?

Here we consider torus link & standard diagram.
Torus link

Fact

The torus link $T(a, b)$ running $a$ times meridionally and $b$ times longitudinally is $\mathbb{Z}$-colorable if $a$ or $b$ is even.

Theorem [I.chihara-M., 2018]

$\text{mincol}_\mathbb{Z}(D) = 4$ for the standard diagram $D$ of $T(pn, n)$ with $n > 2$, even and $p \neq 0$.

The standard diagram of $T(pr, qr)$.

It is known to be a minimal diagram if $p \geq q$. 
Theorem [Ichihara.-Ishikawa-M. (arXiv:1908.00857)]

Let $p, q, r$ be integers such that $p$ and $q$ are coprime with $|p| \geq q \geq 1$, $r \geq 2$. Let $D$ be the standard diagram of $T(pr, qr)$. Suppose that $T(pr, qr)$ is $\mathbb{Z}$-colorable, i.e., $pr$ or $qr$ is even. Then,

$$
\text{mincol}_{\mathbb{Z}}(D) = \begin{cases} 
4 & \text{if } r \text{ is even}, \\
5 & \text{if } r \text{ is odd}.
\end{cases}
$$
Proof of Theorem. [5 colors case: \( r \) is odd]

We assume that the colors are 0, 1, 2, 3 and derive a contradiction. Then, there are only crossings colored as;

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
0 & 2 & 1 & 3 & a & a \\
\end{array}
\]

\( a \in \{0, 1, 2, 3\} \)
Proof of Theorem. [5 colors case: \( r \) is odd]

We assume that the colors are 0, 1, 2, 3 and derive a contradiction. Then, there are only crossings colored as:

\[
\begin{array}{cccc}
0 & 2 & 1 & 3 \\
\hline
1 & 2 & 1 & 3 \\
\hline
& & & a \\
\end{array}
\]

Thus the over arcs must be colored by 1 or 2. And the arcs of a component which has an arc colored by 0 or 2 (1 or 3) are always colored by even (odd) numbers.
Proof of Theorem. [5 colors case: \(r\) is odd, \(q = 1\)]

We may assume that the number of the over arcs colored by 1 is odd in the \(r\) parallel over arcs.
Proof of Theorem. [5 colors case: \( r \) is odd, \( q = 1 \)]

We may assume that the number of the over arcs colored by 1 is odd in the \( r \) parallel over arcs. In the case of \( q = 1 \), since

\[
\exists \text{ an over arc colored by } 0 \Rightarrow \text{ a contradiction.}
\]
Proof of Theorem. [5 colors case: $r$ is odd, $q \geq 2$]

<table>
<thead>
<tr>
<th>0</th>
<th>...</th>
<th>2</th>
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<tr>
<td>2</td>
<td>...</td>
<td>0</td>
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In the case of $q \geq 2$, by going through the $r$ parallel arcs, the color 2 changes to 0, since there are odd over arcs colored by 1.
The change of the colors are expressed by using a linear function $f$. Then we see $f(0) = 2$ and $f(2) = 0$.

That is, $f(X) = -X + 2$. 
Proof of Theorem. \([5 \text{ colors case: } r \text{ is odd, } q \geq 2]\)

From the \(\mathbb{Z}\)-coloring is non-trivial, there exists an arc colored by 3. We see \(f(3) = -1\).

This gives a contradiction. \(\square\)
We can have a complete classification of all the \( \mathbb{Z} \)-colorings on the standard diagram of \( T(pr, qr) \). (Here the details are omitted.)

\[
A = \left\{ (a_1, \ldots, a_q) \in (\mathbb{Z}^r)^q \mid \text{the assignment of } a_1, \ldots, a_q \in \mathbb{Z}^r \text{ to } x_1, \ldots, x_q \text{ defines a } \mathbb{Z} \text{-coloring of } D \right\}
\]

**Proposition 1.**

We have

\[
A = \left\{ \begin{array}{ll}
\{(a, \ldots, a) \mid a \in \mathbb{Z}^r, \Delta(a) = 0\} & \text{if } r \text{ is even,} \\
\{(a, \ldots, a) \mid a \in \mathbb{Z}^r\} & \text{if } r \text{ is odd, } p \text{ is even,} \\
\{(a, \tau(a), a, \ldots, \tau(a)) \mid a \in \mathbb{Z}^r\} & \text{if } r \text{ is odd, } q \text{ is even,}
\end{array} \right.
\]

where \( \Delta(a) = a_1 - a_2 + \cdots + (-1)^r a_r \in \mathbb{Z} \) and \( \tau(a) = (-a_i + 2\Delta(a))_i \in \mathbb{Z}^r \) for \( a = (a_1, \ldots, a_r) \in \mathbb{Z}^r \).
Example: $r$ is even

$T(4n, 8) \ (r = 4), \ a = (0 \ 1 \ 2 \ 1)$
Example: \( r \) is odd, \( q \) is even

\[ T(3n, 6) \ (r = 3, \ q = 2), \ a = (2 \ 1 \ 0), \ \Delta(a) = 1 \]
Results

We can also have a complete classification of all the $\mathbb{Z}$-colorings by only four colors of $T(pr, qr)$. (Here the details are omitted.)

$$A^{(4)} = \left\{ (a_1, \ldots, a_q) \in (\mathbb{Z}^r)^q \ \middle| \ \text{the assignment of } a_1, \ldots, a_q \in \mathbb{Z}^r \text{ to } x_1, \ldots, x_q \text{ defines a } \mathbb{Z}\text{-coloring of } D \right\}$$

Proposition 2.

We have

$$A^{(4)} = \left\{ (a, \ldots, a) \ \middle| \ a \in A^{(4)}_{01} \cup A^{(4)}_{12} \cup A^{(4)}_{23} \right\} \setminus \{(1, \ldots, 1), (2, \ldots, 2)\},$$

where

$$A^{(4)}_{01} = \{(a_1, \ldots, a_r) \in \{0, 1\}^r \ | \ a_1 = a_r = 1, a_{2i} = a_{2i+1} (i = 1, \ldots, r/2 - 1)\},$$

$$A^{(4)}_{12} = \{(a_1, \ldots, a_r) \in \{1, 2\}^r \ | \ a_{2i-1} = a_{2i} (i = 1, \ldots, r/2)\},$$

$$A^{(4)}_{23} = \{(a_1, \ldots, a_r) \in \{2, 3\}^r \ | \ a_1 = a_r = 2, a_{2i} = a_{2i+1} (i = 1, \ldots, r/2 - 1)\}.$$
Example: $A^{(4)}_{01}$

$T(4n, 8) \ (r = 4), \ a = (1 \ 0 \ 0 \ 1)$
Example: $A_{12}^{(4)}$

$T(4n, 8) (r = 4), \ a = (1 \ 1 \ 2 \ 2)$
Thank you for your attention.