Hyperbolicity and the number of components for random link

ランダム絡み目の最頻成分数と双曲性

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In recent Low-dimensional topology, to study 3-manifolds and/or knot via random methods could be a hot topic now.

As a pioneering work, in [1], Dunfield and W.Thurston introduced a random model of 3-manifolds by using random walks on the mapping class group of a surface, and a theory of random 3-manifolds has started. Actually they considered random Heegaard splittings by gluing a pair of handlebodies by the result of a random walk in the mapping class group.

In view of this, the second author introduced and studied two models of random links in [4]. We here report two recent results based on joint works Ichihara-Yoshida [3] and Ichihara-Ma [2].

One model of random link introduced in [4] is defined as the braid closures of the randomly chosen braids via random walks on the braid groups. Suppose that a random walk on the braid group \( \mathfrak{B}_n \) of \( n \)-strings induces a uniform distribution on the symmetric group \( \mathfrak{S}_n \) on \( n \) letters via the natural projection \( \mathfrak{B}_n \to \mathfrak{S}_n \) \((n \geq 3)\). Then, the second author showed in [4, Theorem 1.1] that, for the random link coming from a random walk of \( k \)-step on \( \mathfrak{B}_n \) \((n \geq 3)\), the expected value of the number of components converges to \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \) when \( k \) diverges to \( \infty \).

Then it is natural to ask what is the most expected number of components for such a random link. We first answer to this question as follows in [3].

**Theorem 1** Consider a random link obtained from a random walk on \( \mathfrak{B}_n \). Suppose that the random walk on \( \mathfrak{B}_n \) is defined for the probability distribution on \( \mathfrak{B}_n \) which induces a uniform distribution on \( \mathfrak{S}_n \) via the natural projection \( \mathfrak{B}_n \to \mathfrak{S}_n \) \((n \geq 3)\). Then the most expected number of components is equal to

\[
K_n = \left\lfloor \log(n + 1) + \gamma - 1 + \frac{\zeta(2) - \zeta(3)}{\log(n + 1) + \gamma - 1.5} + \frac{h}{(\log(n + 1) + \gamma - 1.5)^2} \right\rfloor
\]

where \( \lfloor x \rfloor \) denotes the integer part of \( x \), \( \zeta \) the Riemann zeta function, \( \gamma = 0.5772\ldots \) the Euler-Mascheroni constant with \(-1.1 < h < 1.5\). In particular, if \( n > 188 \), it follows that

\[
\left\lfloor \log n - \frac{1}{2} \right\rfloor < K_n < \lfloor \log n \rfloor
\]

Ichihara was partially supported by JSPS KAKENHI Grant Number 26400100. Ma was partially supported by NSFC 11371094. Yoshida was partially supported by JSPS KAKENHI Grant Number 25400050. Ichihara and Yoshida was partially supported by Joint Research Grant of Institute of Natural Sciences at Nihon University for 2015.

2000 Mathematics Subject Classification: Primary 57M25; Secondary 20F36, 60G50.

Keywords: random link, random walk, bridge presentation, hyperbolic.

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In fact, this can be obtained from some results on Combinatorics and Analytic number theory.

To connect the problem of random link to them, the key is the correspondence between components of the closure of a braid and cycles in the cycle decomposition of the permutation corresponding to the braid. In particular, the number of components are calculated as the number of cycles.

In view of this, we can relate random braids and random partitions of integers (the numbers of strings). Then it is also natural to ask what is the most expect partition of the number of strings for a random braid. About this question, against our naive intuition, we can show the following in [3].

**Theorem 2** Consider a random braid obtained from a random walk on $\mathcal{B}_n$. Suppose that the random walk on $\mathcal{B}_n$ is defined for the probability distribution on $\mathcal{B}_n$ which induces a uniform distribution on $\mathfrak{S}_n$ via the natural projection $\mathcal{B}_n \rightarrow \mathfrak{S}_n$ ($n \geq 3$). Then the most expected partition of the number of the strings is $((n - 1), 1)$.

On the other hand, random Heegaard splittings of 3-manifolds were studied extensively by Maher in [6], and, in particular, he showed that a 3-manifold obtained by a random Heegaard splitting is hyperbolic with asymptotic probability 1.

As a natural generalization, as another model of random link, random bridge decomposition for links was considered and studied by the second author in [4].

As a generalization to the result of Maher, we show that a link obtained by a random bridge decomposition is hyperbolic with asymptotic probability 1 as follows in [2].

**Theorem 3** Let $\mathcal{M}_{0,2n}$ be the mapping class group of the $2n$-punctured sphere. For a random walk $\omega_{n,k}$ of length $k$ on $\mathcal{M}_{0,2n}$, the probability that the link admitting a bridge splitting with gluing map $\omega_{n,k}$ is a hyperbolic link in $S^3$ converges to 1 as $k \rightarrow \infty$.

This can be regarded as a dual result of the second author for random braid model in [5]. That is, in [5], he showed that the probability that the link appearing as the braid closure for a random walk on the braid group is hyperbolic converges to 1.

**References**


