ALEXANDER POLYNOMIALS OF DOUBLY PRIMITIVE KNOTS

KAZUHIRO ICHIHARA, TOSHIO SAITO, AND MASAKAZU TERAGAITO

1. Motivation

Given a knot $K$ in the 3-sphere $S^3$ and an irreducible fraction $p/q$, one can construct a closed 3-manifold as follows: Removing an open tubular neighborhood of $K$ from $S^3$, and then gluing a solid torus back as its meridian is identified with the $(p, q)$-curve on the peripheral torus of $K$. This operation is called Dehn surgery, and gives an important subject in the study of knots and 3-manifolds. See [3] for a survey.

It is experimentally observed that most Dehn surgery on a “complicated” knot tend to make “complicated” manifolds. For example, Dehn surgery on the trivial knot gives only “simple” 3-manifolds, called lens spaces. Thus it seems to be a natural question: Which non-trivial knots admit Dehn surgery yielding Lens space? A lot of results have been achieved to answer this, but is still remaining open.

Among the works in this streaming, that of John Berge [1] should be featured prominently, though it has not been published. He introduced the concept of doubly primitive knots, and showed that they all admit such Dehn surgery. In fact, C. Gordon presently conjectures that only these knots admit such surgery. See [3] or [7] for details.

In the study toward the conjecture, recently, several researcher groups, P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó [8], P. Ozsváth and Z. Szabó [9], T. Kadokami [5], T. Kadokami and Y. Yamada [6], found that the knots with Dehn surgery yielding lens spaces have some restrictions on their Alexander polynomials.

In view of their results, toward the Gordon’s conjecture, it would be worth to study the Alexander polynomials of doubly primitive knots. The aim of this manuscript is to introduce the formula which we have obtained to calculate it. Please see our preprint [4] for full details.

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2. Formula

We start with definitions of the doubly primitive knot and the Alexander polynomial.

A knot $K$ in $S^3$ is called doubly primitive if it lies on the standard genus two Heegaard surface $S$ of $S^3$, and it represents primitive elements in both fundamental groups of the two handlebodies bounded by $S$. This can be defined topologically in the following way. A knot $K$ in $S^3$ is called doubly primitive if it lies on the standard genus two Heegaard surface $S$ of $S^3$, and there exist a pair of meridian disks in each handlebody bounded by $S$, one of which is disjoint from $K$, whereas the other intersects $K$ in a single point.

For example, the trefoil is shown to be doubly primitive; as you can find such disks in the following figure (Exercise).

The Alexander polynomial of a knot is one of the most well-known classical knot invariant. Here we give its definition very roughly as follows. See [2] for more details in example.

For a knot $K$ in $S^3$, let $G_K$ be the knot group $\pi_1(S^3 - K)$. Assume that $G_K$ has a presentation $\langle x_1, \ldots, x_u \mid r_1, \ldots, r_v \rangle$. Then the $u \times v$-matrix $A_K = \left( \hat{\alpha} \circ \hat{\phi} \left( \frac{\partial x_i}{\partial x_j} \right) \right)$ is called the Alexander matrix, and the polynomial determined by the greatest common divisor of the $v \times v$-minors of $A_K$ is called the Alexander polynomial of $K$. Here, $\hat{\alpha} : ZG_K \to Z[t]$ and $\hat{\phi} :ZF_u \to ZG_K$ denotes the induced homomorphisms on the group rings from the abelianization $\alpha : G_K \to Z$ and the canonical homomorphism $\phi : F_u \to G_K$, respectively. And $\frac{\partial}{\partial x_j} : ZF_u \to ZF_u$ stands for the so-called Fox’s free differential.

Next, we have to explain our “parameterization” of doubly primitive knots.
Suppose that a doubly primitive knot $K$ admits Dehn surgery yielding the lens space $L(p, q)$. Then let $K^*$ be the dual knot of $K$, i.e., the knot appearing as the core curve of the attached solid torus. This knot $K^*$ can be isotoped in a 1-bridge braid position in $L(p, q)$, by virtue of the theorem of Berge, and, in particular, the position is determined by a triple of integers, say, $(p, q, k)$. Here we omit the precise definitions about this. Please refer [10] for example. We insist that such triples can be obtained from some information of the diagram on the Heegaard surface where the knot is in a doubly primitive position.

We need to prepare two numerical functions to state our result. For a given triple $(p, q, k)$ and a number $i \in \{0, 1, 2, \ldots, p - 1\}$, we set $\psi(i)$ as the unique number determined by

$$\psi(i) \cdot q \equiv i \pmod{p}, \quad 1 \leq \psi(i) \leq p$$

and set

$$\phi(i) = \# \{ j \mid \psi(j) < \psi(i), \quad 1 \leq j \leq k - 1 \}$$

where $\#$ indicates the cardinality of the set. This can be easily seen in the following way. Consider the sequence $\{ qn \pmod{p} \}_{n=1}^p$, and then, $\psi(i)$ expresses the position of $i$ in the sequence and $\phi(i)$ the number of terms smaller than $k$ before appearing $i$.

Now we can state our main theorem:

**Theorem.** Let $K$ be a doubly primitive knot in $S^3$. Suppose that Dehn surgery on $K$ yields $L(p, q)$, and the dual knot $K^*$ in $L(p, q)$ is parametrized by $(p, q, k)$. Then the Alexander polynomial $\Delta_K(t)$ of $K$ is presented as

$$\left( \sum_{i=0}^{k-1} t^{\phi(i)p-\psi(i)k} \right) / \left( t^{k-1} + t^{k-2} + \cdots + t + 1 \right)$$

up to multiplication by a unit $\pm t^n$.

As a byproduct, we get a practical algorithm to determine the genus of a doubly primitive knot. Because every doubly primitive knot is shown to be fibered in [9], and then, it is well-known that the genus of a fibered knot is determined by the Alexander polynomial.

Also, by using the proof of the theorem, we recovers a result in [9]. That is the restriction on the form of Alexander polynomials for doubly primitive knots.

The proof of the theorem can be established as follows. For a given doubly primitive knot, we start with the presentation of the knot group which is described by the parameter of dual knot. This was used in [9] to show that doubly primitive knots are all fibered. In standard way, by Fox’s free calculus, we get the Alexander matrix, which is in fact
1 × 2-matrix. The entries are two Laurent polynomials including three parameters. The hardest and most technical part of our proof is to take their greatest common divisor. Actually, before finding the correct proof, we had to have a number of trial and errors. It was a great help to do a lot of computer-aided-experiments, and have observed the behavior of the polynomials. Please see our preprint [4] for detailed calculations.

3. Examples

We here include two examples.

First one is the trefoil knot. As is well-known, 5-surgery on it yields the lens space $L(5, 4)$. It is also shown that the dual knot has type $(p, q, k) = (5, 4, 2)$. Consider the sequence:

$$\{4n \pmod{5}\}_{n=1}^{5} = \{4, 3, 2, 1, 0\}$$

Then we get the following table:

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(i)$</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$\phi(i)$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\phi(i)p - \psi(i)k$</td>
<td>$-5$</td>
<td>$-8$</td>
</tr>
</tbody>
</table>

From this table, we obtain the Alexander polynomial as:

$$\frac{t^{-5} + t^{-8}}{t + 1} = t^{-8}(t^2 - t + 1).$$

Next example is the $(-2, 3, 7)$-pretzel knot. It is famous that 18-surgery on the knot creates the lens space $L(18, 5)$. It can be checked that the dual knot has type $(p, q, k) = (18, 5, 7)$. Thus we have the following sequence and table:

$$\{nq \pmod{18}\}_{n=1}^{18} = \{5, 10, 15, 2, 7, 12, 17, 4, 9, 14, 1, 6, 11, 16, 3, 8, 13, 0\}$$

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(i)$</td>
<td>18</td>
<td>11</td>
<td>4</td>
<td>15</td>
<td>8</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>$\phi(i)$</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$\phi(i)p - \psi(i)k$</td>
<td>$-18$</td>
<td>$-23$</td>
<td>$-10$</td>
<td>$-15$</td>
<td>$-20$</td>
<td>$-7$</td>
<td>$-12$</td>
</tr>
</tbody>
</table>

Consequently we get the Alexander polynomials as follows.

$$\frac{t^{-18} + t^{-23} + t^{-10} + t^{-15} + t^{-20} + t^{-7} + t^{-12}}{t^6 + t^5 + t^4 + t^3 + t^2 + t + 1}$$

$$= t^{-23}(1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10})$$
References


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