EUCLIDEAN LENGTH ON A HOROTORUS
AND THE CULLER-SHALEN NORM OF SLOPES

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Abstract. In the study of exceptional Dehn fillings, two functions of slopes, called the Euclidean length on a horotorus and the Culler Shalen norm, are playing important roles. In this paper, we investigate their relationship, and give two inequalities between them.

1. Introduction

The aim of this manuscript is to report the recent results given by the author, which concern the relationships between the Euclidean length on a horotorus and the Culler-Shalen norm of slopes. Also included are reports on the study of a particular example and computer-aided experiments. The former would exhibit the efficiency of our results and the latter would suggest certain direction of further studies.

Throughout this manuscript, $M$ denotes a compact, connected, orientable 3-manifold with single torus boundary $\partial M$. We always suppose that $M$ is hyperbolic, i.e. the interior of $M$ admits a complete Riemannian metric with constant sectional curvature $-1$.

The main object considered here is the set of slopes on $\partial M$: A slope on $\partial M$ is defined as the isotopy class of an unoriented non-trivial simple closed curve on $\partial M$.

2. Background

The famous Geometrization Conjecture raised by Thurston [11] says that: All compact 3-manifolds are classified as; reducible, toroidal, Seifert fibered, or hyperbolic ones. This would gives a complete classification of all 3-manifolds.

Date: November 24, 2005.

2000 Mathematics Subject Classification. Primary 57M50; Secondary 57M25.

Key words and phrases. slope, Culler-Shalen norm, Exceptional surgery, boundary slope.

Report manuscript for the Proceeding of “Intelligence of Low Dimensional Topology” (2005.11.7–11.10, the Umeda Satellite of Osaka City University).
Toward further studies, one of the next direction in 3-manifold theory is to study the relationships between 3-manifolds. One of the important operation describing a relationship between 3-manifolds is the *Dehn surgery*. That is an operation to create a new 3-manifold from a given 3-manifold and a given knot (i.e., an embedded simple closed curve) as follows: Take an open tubular neighborhood of the knot, remove it, and glue a solid torus back in a different manner. This gives an interesting subject to study; for example, it is shown that any pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries [8].

The last step of the Dehn surgery is particularly called a *Dehn filling* and studied in detail independently. Precisely that is an operation to glue a solid torus $V$ to $M$. One of the motivation to study the Dehn filling lies in the following fact due to Thurston [10]: If $M$ is hyperbolic, then all but finitely many Dehn fillings yield hyperbolic 3-manifolds. In view of this, the finitely many exceptions are called *exceptional fillings*, and it seems very natural to ask:

**Question.** When, how many, what kind of exceptional fillings can occur?

Toward answering this question, two functions on the set of slopes on $\partial M$ had been developed and played important roles. Those are, the Euclidean length $L_T(\cdot)$ on a horotorus $T$ and the Culler-Shalen norm $\| \cdot \|$. For example, we have the following excellent results. In the following, $M(r)$ denotes the 3-manifold obtained by Dehn filling such that the meridian of $V$ is glued to a curve representing the slope $r$ on $\partial M$. For, such a slope determines the homeomorphism type of the resultant manifold.

- If $r$ is not a strict boundary slope and $\pi_1(M(r))$ is cyclic, then $\|r\|$ takes the least value among non-trivial slopes [4].
- If $r$ is not a strict boundary slope and $\pi_1(M(r))$ is finite, then $\|r\|$ is at most 3 times of the least value among non-trivial slopes [3].
- If $M(r)$ does not admit a negatively-curved Riemannian metric, then $L_T(r)$ is at most $2\pi$ [2].
- If $M(r)$ is reducible or $\pi_1(M(r))$ is not word-hyperbolic, then $L_T(r)$ is at most 6 [1, 6].

The definitions of $L_T(\cdot)$ and $\| \cdot \|$ are quite different, but all the theorem above seems to have somehow similar flavors: If $M(r)$ is non-hyperbolic, then the value of $r$ is relatively small. Motivated by this, we will consider the next question:

**Question.** Is there a relationship between $L_T(\cdot)$ and $\| \cdot \|$?
3. Definitions and Results

Before stating our results, we give very rough definitions of the Euclidean length $L_T(\cdot)$ on a horotorus $T$ and the Culler-Shalen norm $\| \cdot \|$.

3.1. Length on a horotorus. We are assuming that the interior of $M$ is a complete hyperbolic 3-manifold of finite volume. This implies its universal cover can be identified with $\mathbb{H}^3 = \left( \mathbb{R}^3_+, \frac{dx^2+dy^2+dz^2}{z^2} \right)$, which is a simply connected, open Riemannian 3-manifold of constant curvature $-1$, called the hyperbolic 3-space.

The fundamental group $\pi_1(M)$ acts on $\mathbb{H}^3$ as covering transformations. Under this action, a horizontal plane $\{ z = const \} \subset \mathbb{H}^3$, called a horosphere, is shown to be equivariant if $z$ is sufficient large.

On the half space lower bounded by such a horosphere, the action of $\pi_1(M)$ is a product action. Thus we obtain an embedded boundary-parallel torus in $M$ by covering projection, which is called a horotorus.

The induced metric on a horosphere in $\mathbb{H}^3$ is a Euclidean metric. Since the covering map is local isometry, we can find a Euclidean metric on a horotorus $T$ in $M$. Thus the length of a loop on a horotorus is defined. By identifying $\partial M$ and $T$, the length of a slope on $\partial M$ is defined as the minimal length of the representatives on $T$.

We here recall that the hyperbolic structure on the interior of $M$ uniquely derives a particular $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(M)$.

The group $\pi_1(M)$ is regarded as the covering transformation groups, and so, it is identified with a discrete subgroup of $\text{Isom}^+(\mathbb{H}^3)$; the group of orientation preserving isometry of $\mathbb{H}^3$. Using the Identification of the ideal boundary $\partial \mathbb{H}$ with $\mathbb{C} \cup \{ \infty \}$, $\text{Isom}^+(\mathbb{H}^3)$ is identified with $\left\{ \frac{px+q}{rx+s} \subset \mathbb{C} \right\} / \pm I$. And this is also regarded as $\text{PSL}_2(\mathbb{C})$. Thus we obtain a faithful discrete representation; $\pi_1(M) \rightarrow \text{PSL}_2(\mathbb{C})$. This can be lifted to $\pi_1(M) \rightarrow \text{SL}_2(\mathbb{C})$, which we call the holonomy representation, denote by $\rho_{\text{hol}}$, in this manuscript.

3.2. The Culler-Shalen norm. For a representation $\rho : \pi_1(M) \rightarrow \text{SL}_2(\mathbb{C})$, the character $\chi_\rho : \pi_1(M) \rightarrow \mathbb{C}$ is defined by $\chi_\rho(\gamma) := \text{trace}(\rho(\gamma))$ for $\gamma \in \pi_1(M)$. The set of characters for all $\text{SL}_2(\mathbb{C})$-representations of $\pi_1(M)$, after taking the closure and doing desingularizations, is called the character variety, denoted by $X(\pi_1(M))$.

The principal component $X_0(\pi_1(M))$ of $X(\pi_1(M))$ is the irreducible component including the character of $\rho_{\text{hol}}$. It is known that $\dim_{\mathbb{C}} X_0(\pi_1(M)) = 1$, i.e., $X_0(\pi_1(M))$ gives an algebraic curve, or, topologically it is a compact 2-manifold.
For a slope \( r \) on \( \partial M \), we then have a rational map \( I_r : X_0(\pi_1(M)) \rightarrow \mathbb{C}P^1 \) defined by \( I_r(\chi \rho) := \chi \rho(\gamma) = \text{trace}(\rho(\gamma)) \). Here \( \gamma \) denotes the element of \( \pi_1(M) \) represented by a curve of slope \( r \). As such a \( \gamma \) is determined up to conjugacy, \( I_r \) is well-defined.

Note that \( I_r : X_0(\pi_1(M)) \rightarrow \mathbb{C}P^1 \) is topologically regarded as a branched covering map of a closed surface over the 2-sphere. Then, for a slope \( r \) on \( \partial M \), the Culler-Shalen norm is defined by \( 2\deg(I_r) \), which we denote by \( \| r \| \).

### 3.3. Results.
Concerning the relationship between \( \| \cdot \| \) and \( L_T(\cdot) \), we obtain the following inequalities.

**Theorem 1.** Suppose that \( M \) is the exterior of a hyperbolic two-bridge knot, or a \((-2, 3, n)\)-pretzel knot for odd \( n \geq 7 \) in the 3-sphere. Then

\[
\| r \| \geq \frac{2}{3} L_T(r)
\]

holds for any slope \( r \) on \( \partial M \) and for any horotorus \( T \).

**Theorem 2.** Let \( m \) be the fixed meridional slope, and \( r_1, r_2 \) integral slopes on \( \partial M \). Suppose that \( m \) is not a strict boundary slope. If \( r_1 \) is greater than or equal to the maximal boundary slope and \( r_2 \) is less than or equal to the minimal boundary slope for \( M \), then

\[
L_T(r_1) + L_T(r_2) > \frac{\| r_1 \|}{\| m \|} + \frac{\| r_2 \|}{\| m \|}
\]

holds for the maximal horotorus \( T \).

We remark that there exists no universal constant \( C \) such that \( L_T(r) > C \| r \| \) holds in general.

We here omit the precise definitions of terms appearing in the statement of Theorem 2, except for the following.

**Definition** (Boundary slope). A slope on \( \partial M \) determined by the boundary components of an essential embedded surface in \( M \) is called the boundary slope.

Note that the Culler-Shalen theory detects certain boundary slopes.

### 4. Example and Experiments

In the last section, we will give an outline of the proof of Theorem 1. Before that, in this section, we will first give a description of the Euclidean length on the maximal horotorus and the Culler-Shalen norm of slopes for a particular example. This would show how our inequalities are effective. Second, we will report on computer-aided
experiments about the Culler-Shalen norm of boundary slopes. This would suggest another direction for further study.

4.1. Example: the figure-eight knot. Let $M$ be the exterior of the figure-eight knot in the 3-sphere $S^3$. This is well-known to be hyperbolic. See [10] for example. We take the maximal horotorus $T$.

![Figure 1. the figure-eight knot](image)

In [10], the modulus of the maximal horotorus is investigated in detail. In fact, on the universal cover of $T$, we have the following diagram.

![Figure 2](image)

In the figure, $\mu, \lambda$ denote the lifts of the meridian, longitude, with suitable orientations. It is shown that $\mu$ and $\lambda$ are perpendicular each other, and $L_T(\mu) = 1$.
and $L_T(\lambda) = 2\sqrt{3}$. Thus, for example, it is calculated as $L_T(r_{(1,4)}) = 2\sqrt{7}$ for the slope $r_{(1,4)}$, which runs once in the longitudinal direction and 4 times in the meridional direction.

On the other hand, by virtue of the result in [4], the Culler-Shalen norm can be calculated as follows.

**Proposition 1.** Let $s_1, \cdots, s_n$ be boundary slopes for $\partial M$. Then there exist non-negative even constants $a_1, \cdots, a_n$ such that $\|r\|$ is expressed as $\sum_{i=1}^{n} a_i \Delta(r, s_i)$ for any slope $r$ on $\partial M$. Moreover, at least two $a_i$’s are non-zero.

It is known that if $s_i$ can be detected by the Culler-Shalen theory, then the coefficient $a_i$ is non-zero. Here the distance $\Delta(r, r')$ of two slopes $r, r'$ is the minimal geometric intersection number of the representatives of $r, r'$.

For the figure-eight knot exterior case, from the Ohtsuki’s result [9], we have $\|r\| = 2\Delta(r, r_{(1,4)}) + 2\Delta(r, r_{(1,-4)})$. Thus, we obtain

$$\frac{8}{\sqrt{3}} L_T(r) \geq \|r\| \geq \frac{8}{\sqrt{7}} L_T(r) .$$

Please compare this with the inequality given by Theorem 1. It would suggest that the bound given in Theorem 1 can be improved.

Moreover, in this case, $\|m\| = 4$. So we have;

$$L_T(r) \geq \frac{\sqrt{3}}{2} \frac{\|r\|}{\|m\|} .$$

The equality holds for the longitude $r_{(1,0)}$.

In particular, for $r_{(n,1)}$ with $n \geq 4$ or $n \leq -4$, we have

$$L_T(r_{(n,1)}) > \frac{\|r_{(n,1)}\|}{\|m\|} .$$

Please compare this with the inequality given by Theorem 2. This shows that the inequality given in Theorem 2 is optimal in a sense.

4.2. **Computer Experiments.** This part is supported by Shigeru Mizushima (Tokyo Institute of Technology).

In the previous subsection, we have observed that there exists certain relationship between $L_T(r)$ and $\frac{\|r\|}{\|m\|}$.

On the other hand, it is known that

**Proposition 2 ([1, 6]).** For an essential surface $F$ in $M$ with boundary slope $r$, $6 - \chi(F) > 20_F \geq L_T(r)$ holds.
Thus we can ask:

**Question.** Is there a relationship between $\frac{x(F)}{\partial F}$ and $\frac{||r||}{|m|}$ for a boundary slope $r$?

Concerning this question, we performed a computer-aided experiments about boundary slopes for two-bridge knots in $S^3$.

We used Dunfield’s computer program [5] to find and list up all boundary slopes of essential surfaces for two-bridge knots, and Ohtsuki’s formula [9] (corrected by Mattman [7]) to compute their Culler-Shalen norm.

As is well-known, the two-bridge knots in $S^3$ are parametrized by irreducible fractions. The range of our experiments are the fractions with denominators and numerators up to 100, 200, 300, 400, and we obtained the following graphs.

These seem to suggest there exists certain (linear?) relationship between them.
5. Outline of Proof (Theorem 1)

Actually we show the following theorem.

**Theorem 3.** Suppose that there exists two essential surfaces $S_1, S_2$ in $M$ with boundary slopes $s_1, s_2$ such that

$$
\Delta(s_1, s_2) \geq 2 - \frac{\chi(S_i)}{\partial S_i}
$$

holds for $i = 1, 2$. Suppose further that $s_1, s_2$ are detected by the Culler-Shalen theory. Then $\|r\| \geq \frac{2}{3} L_T(r)$ holds for any slope $r$ on $\partial M$ and for any horotorus $T$.

For a hyperbolic two-bridge knot exterior and a $(-2, 3, n)$-pretzel knot exterior ($n \geq 7$, odd), it is shown that the assumption of Theorem 3 is satisfied by using the result of Ohtsuki [9] and Mattman [7]. Thus Theorem 3 implies Theorem 1.

We remark that:

- The boundary slopes undetected by the Culler-Shalen theory are found very recently,
- An alternating knot exterior always contains two essential surfaces $S_1, S_2$ satisfying $\Delta(s_1, s_2) \geq 2 - \frac{\chi(S_i)}{\partial S_i}$. They are given by the checker-board surfaces for the irreducible alternating diagram. However it is unknown whether the boundary slopes are detected or not.

To prove Theorem 3, we prepare the following lemma.

**Lemma.** On any horotorus $T$, we have

$$
\Delta(r, r') = \frac{L_T(r) \cdot L_T(r') \cdot \sin |\theta_r - \theta_{r'}|}{\text{Area}(T)},
$$

where $\theta_r$ denotes the angle between the geodesic representatives of $r$ and $m$ on $T$. $\Box$

**Outline of Proof of Theorem 3.** Let $s_1, \ldots, s_n$ be boundary slopes on $\partial M$.

By Proposition 1, we have $\|r\| = \sum_{i=1}^n a_i \Delta(r, s_i)$, and by assumption, there exist two boundary slopes, say $s_1, s_2$, on $\partial M$ detected by the Culler-Shalen theory, i.e., $a_1, a_2$ are non-zero.

Then we have the following using Lemma 1.

$$
\|r\| \geq a_1 \Delta(r, s_1) + a_2 \Delta(r, s_2) \\
= \frac{L_T(r) \cdot L_T(s_1) \cdot \sin |\theta_r - \theta_{s_1}|}{\text{Area}(T)} + \frac{L_T(r) \cdot L_T(s_2) \cdot \sin |\theta_r - \theta_{s_2}|}{\text{Area}(T)} \\
= \frac{L_T(r)}{\text{Area}(T)} (a_1 L_T(s_1) \sin |\theta_r - \theta_{s_1}| + a_2 L_T(s_2) \sin |\theta_r - \theta_{s_2}|).$
$$
Now, without loss of generality, we assume that $a_1 L_T(s_1) \geq a_2 L_T(s_2)$. Then we obtain:

\[
\|r\| \geq \frac{L_T(r)}{\text{Area}(T)} \cdot a_2 L_T(s_2) \cdot (\sin |\theta_r - \theta_{s_1}| + \sin |\theta_r - \theta_{s_2}|)
\]

By using the elementary Euclidean trigonometry, we claim that $\sin |\theta_r - \theta_{s_1}| + \sin |\theta_r - \theta_{s_2}| > \sin |\theta_{s_1} - \theta_{s_2}|$ holds.

It follows from this claim and Equation (1)

\[
\|r\| > \frac{L_T(r)}{\text{Area}(T)} \cdot a_2 L_T(s_2) \cdot \sin |\theta_{s_1} - \theta_{s_2}|
\]

\[
= a_2 \cdot \frac{L_T(r) \cdot L_T(s_1) L_T(s_2) \sin |\theta_{s_1} - \theta_{s_2}|}{L_T(s_1)} \cdot \text{Area}(T)
\]

\[
= a_2 \cdot \frac{L_T(r) \cdot \Delta(s_1, s_2)}{L_T(s_1)}.
\]

By Proposition 2, we obtain

\[
\|r\| \geq a_2 \cdot \frac{L_T(r) \cdot \Delta(s_1, s_2)}{L_T(s_1)} \geq \frac{a_2}{6} \cdot \frac{\Delta(s_1, s_2)}{-\chi(S_1)/2 \partial S_1} \cdot L_T(r)
\]

By the assumption that $\Delta(s_1, s_2) \geq 2 - \chi(S_1) / 2 \partial S_1$ and $a_2 \geq 2$, we conclude that

\[
\|r\| \geq 2 \cdot \frac{a_2}{3} \cdot L_T(r).
\]

\[
\square
\]

REFERENCES

5. N. Dunfield, computer program, freely available at http://www.its.caltech.edu/~dunfield/


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