CROSSCAP NUMBERS OF PRETZEL KNOTS

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Abstract. For a non-trivial knot in the 3-sphere, the crosscap number is defined as the minimal first betti number of non-orientable spanning surfaces for it. In this article, we report a simple formula of the crosscap number for pretzel knots.

1. Introduction

It is well-known that any knot in the 3-sphere $S^3$ bounds an orientable subsurface in $S^3$; called a Seifert surface. One of the most basic invariant in knot theory, the genus of a knot $K$, is defined to be the minimal genus of a Seifert surface for $K$.

On the other hand, any knot in $S^3$ also bounds a non-orientable subsurface in $S^3$: Consider checkerboard surfaces for a diagram of the knot, one of which is shown to be non-orientable. In view of this, similarly as the genus of a knot, B.E. Clark defined the crosscap number of a knot as follows.

Definition (Clark, [1]). The crosscap number $\gamma(K)$ of a knot $K$ is defined to be the minimal first betti number of a non-orientable surface spanning $K$ in $S^3$.

For completeness we define $\gamma(K) = 0$ if and only if $K$ is the unknot.

Example. The figure-eight knot $K$ in $S^3$ bounds a once-punctured Klein bottle $S$, appearing as a checkerboard surface for the diagram of $K$ with minimal crossings. Thus $\gamma(K) \leq \beta_1(S) = 2$. In fact, it is shown that $\gamma(K) = 2$ by the following proposition together with the well-known fact that $K$ is not cabled.

Proposition 1 (Clark). $\gamma(K) = 1$ if and only if $K$ is a $(2, n)$-cabled knot.
In general, it is rather difficult to determine the crosscap number of a given knot. Here we collect known results about the determination of the crosscap number of knots.

1. In [6], Murakami and Yasuhara determined the crosscap number of the knot 7_4 in the knot table, that is 3: \( \gamma(7_4) = 3 \). This would be the first example for which the crosscap number is determined to be greater than two.

2. In [7], Teragaito gave a formula for the crosscap numbers of torus knots. It is described by using a recursive formula defined in [2].

3. In [4], Hirasawa and Teragaito established an algorithm for the crosscap numbers of the two-bridge knots. The algorithm is efficient and practical.

2. Results

In this article, we report our result on the crosscap number of pretzel knots. Actually we determine the crosscap number completely for them. Let \( K \) be a pretzel knot \( P(p_1, p_2, \ldots, p_n) \). That is, \( K \) is composed from a number of \( 1/p_i \)-tangles in line.

It should be remarked that \( K \) become a knot (not a link) if and only if

(a) some \( p_i \) is even, and the others are odd, or,

(b) \( n \) is odd, and \( p_1, p_2, \ldots, p_n \) are all odd.

Observe that, in the case where \( K \) is of (type a), \( \gamma(K) \leq n - 1 \) holds; consider naturally spanned non-orientable surface. (See Figure 1 left.) On the other hand, in the case where \( K \) is of (type b), \( \gamma(K) \leq n \) holds; consider naturally spanned Seifert surface, and adding a small half-twist. (See Figure 1 right.)

Our theorem says that the equalities above actually hold always.

**Theorem.** Let \( K \) be a pretzel knot \( P(p_1, \ldots, p_n) \), where the number of tangles \( n \geq 2 \). Then we have

\[
\gamma(K) = \begin{cases} 
  n - 1 & \text{if } K \text{ is of (type a)} \\
  n & \text{if } K \text{ is of (type b)}
\end{cases}
\]

Please note that, if \( n = 2 \), then \( K \) is \((2,k)\)-cabled, and \( \gamma(K) = 1 \), and, if \( n = 1 \), then \( K \) is trivial, and \( \gamma(K) = 0 \).

Also remark that, if \( K \) is of (type b), then \( \gamma(K) = 2g(K) + 1 \) holds, where \( g(K) \) denotes the genus of the knot.
Let $K$ be a pretzel knot. Actually we prove:

**Proposition 2.** Any essential surface $F$ for $K$ satisfies

\[
\frac{-\chi}{\sharp} \geq \begin{cases} 
    n - 3 & \text{(type a)} \\
    n - 2 & \text{(type b)}
\end{cases}
\]

Here $\chi$ denotes the Euler characteristic of $F$ and $\sharp$ is the number of sheets of $F$, i.e., the minimal number of the intersection of $\partial F$ and the meridian of $K$. Moreover, if the equality holds, $F$ fails to be a non-orientable spanning surface.

This proposition implies the theorem as follows. Suppose that a non-orientable spanning surface $F$ attaining $\gamma(K)$ is given. If $F$ is essential, with simple calculations, Proposition 2 directly assures that Inequality (1) holds. Otherwise, by boundary-compression, we have an essential surface $F'$ from $F$ with smaller $-\chi$. Then Proposition 2 assures that Inequality (1) holds also in this case.

4. **Boundary slopes for Montesinos knots**

The key to prove Proposition 2 is the Hatcher - Oertel’s algorithm given in [3]. In fact, they gave an algorithm to list up all boundary slopes for a given Montesinos knot.

In their algorithm, any essential surface $F$ in the exterior $E(K)$ of a Montesinos knot $K$ corresponds to a set of edgepaths in the diagram $\mathcal{D}$ illustrated in Figure 2.
We call such a set of edgepaths the edgepath system corresponding to $F$. From an edgepath systems satisfying certain conditions, one can construct a properly embedded surface in $E(K)$, called a candidate surface. In general, candidate surfaces can be inessential, but we can actually show that Inequality (1) holds for all candidate surfaces.

For candidate surfaces, there exist formulae to calculate $\frac{\pi}{e}$, implicitly given in [3]. By using their formulae we can prove Proposition 2 as follows. Recall that candidate surfaces are classified into type I, II, or III.

For type II and III surfaces, by using the formulae, it is easy to check Inequality (1) holds.

For type I case, by the formula, it can be checked Inequality (1) holds except for the following special cases: $P(-2, 3, 3), P(-2, 3, 5), P(3, 3, n), P(3, 5, 5)$. These cases can be studied individually by analyzing the possible shape of edgepath systems. Then we can see that Inequality (1) holds also in these cases.

The main technique we used was almost developed in our previous paper [5]. Please see it for more detailed explanations and arguments.

REFERENCES


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