In this talk, we consider a 2-dimensional cellular complex geodesically embedded in a hyperbolic 3-manifold \( M \), and study its area defined as the sum of the area of its 2-cells. Such an object, for instance, is obtained from the boundary of a Dirichlet polyhedron for \( M \) closed. When \( M \) has cusps or totally geodesic boundary, another example is given by the cut locus. Both are often used to study hyperbolic 3-manifolds and kleinian groups, and so, our attention will be focused to these cellular complexes.

First we consider the case that \( M \) is a closed hyperbolic 3-manifold. As usual, we identify the universal cover of \( M \) with the 3-dimensional hyperbolic space \( \mathbb{H}^3 \). Let \( \Gamma \) be the covering transformation group. Fix a point \( x \) in \( M \) and a lift \( \tilde{x} \) of \( x \) in \( \mathbb{H}^3 \). Then the Dirichlet polyhedron \( D_x \) of \( M \) (with center \( \tilde{x} \)) is defined as the set of points in \( \mathbb{H}^3 \) closer to \( \tilde{x} \) than \( \gamma \cdot \tilde{x} \) for any \( \gamma \in \Gamma \). This becomes a convex fundamental polyhedron for \( \Gamma \) with finite number of totally geodesic sides (see [5] for example). A Dirichlet polyhedron \( D_x \) is called generic if the dual to the decomposition of \( \mathbb{H}^3 \) obtained from all translates of \( \partial D_x \) by \( \Gamma \) is a triangulation. Then our first theorem is the following:

**Theorem 1.** Let \( M \) be a closed hyperbolic 3-manifold and \( D_x \) a Dirichlet polyhedron of \( M \). Suppose that \( D_x \) is generic. Then \( \text{Area}(\partial D_x) < 2\pi(v/4 - 2) \) holds, where \( v \) denotes the number of vertices of \( D_x \).

Remark that it was shown in [3] that Dirichlet polyhedra are generic for almost all points in \( M \).

The following is an immediate corollary of the theorem above.

**Corollary 1.** Every generic Dirichlet polyhedron of a closed hyperbolic 3-manifold has at least twelve vertices.

**Proof.** Since \( \text{Area}(\partial D_x) \) is positive, the number \( v \) of vertices of \( D_x \) is greater than eight. On the other hand, each four vertices of \( D_x \) are glued together in \( M \) by covering projection, and so \( v \) is a multiple of four. \( \square \)

By definition, we also obtain the following immediately.

**Corollary 2.** For every point \( x \) in a closed hyperbolic 3-manifold, there exist more than two points each of which is connected by four distinct shortest geodesic segments to \( x \). \( \square \)
The existence of such points in a compact, non-positively curved Riemannian manifold is known [2] and generically these are only finite many. While there exist uncountably many points each of which is connected by at most three distinct geodesic segments to the given point.

Next we consider the case that $M$ is complete and has finite volume. We take a set of mutually disjoint holotori $T$ in $M$ which bound neighborhoods of all cusps. The cut locus $C$ (with respect to $T$) is defined as the set of points in $M$ admitting at least two distinct shortest paths to $T$. By definition, obviously, $C$ is a geodesic, convex 2-dimensional cellular complex embedded in $M$. The dual to $C$ yields a Euclidean decomposition of $M$ defined in[1]. Moreover, when we take holotori which bound the cusp neighborhoods of the same volume, the corresponding $C$ equals the image of the boundary of the Ford region for $M$ by the universal covering projection, which is called a Ford complex. In this case, we obtain:

**Theorem 2.** Let $M$ be a complete, non-compact hyperbolic 3-manifold of finite volume without boundary. Let $T$ be a set of mutually disjoint holotori in $M$ which bound neighborhoods of all cusps and $C$ the cut locus with respect to $T$. Suppose that the Euclidean decomposition of $M$ dual to $C$ consists of $t$ ideal tetrahedra. Then $\text{Area}(C) < \pi t$ holds.

In the case that $M$ is compact and has non-empty totally geodesic boundary $\partial M$, the cut locus $C$ (with respect to $\partial M$) is defined in a similar way as above: $C$ consists of points in $M$ admitting at least two distinct shortest paths to $\partial M$. The textitcanonical decomposition of $M$ is defined as the geometric dual to $C$ [4]. Again this $C$ is a geodesic, convex 2-dimensional cellular complex embedded in $M$ [4], and we obtain:

**Theorem 3.** Let $M$ be a compact hyperbolic 3-manifold with non-empty totally geodesic boundary $\partial M$ and $C$ the cut locus with respect to $\partial M$. Suppose that the canonical decomposition of $M$ consists of $t$ truncated tetrahedra. Then $\text{Area}(C) < t\pi + 1/2\text{Area}(\partial M)$ holds. Moreover $\text{Area}(C) < 3\pi t$ holds.

**References**