Theorem:
$M$: closed orientable 3-manifold
$F$: a Heegaard surface of genus $g$,
$b, n$: any positive integers
Then
$\exists K$: a knot in $M$
with a $(g, b)$-bridge splitting of distance $> n$ w.r.t. $F$
except for $(g, b) = (0, 1), (0, 2)$.

1. Strategy of Proof

Suppose: $M$ admits a Heegaard surface $F$
i.e., $\exists$ homeo. $M \rightarrow V_1 \cup_f V_2$ s.t. $f(F) = \partial V_1 = \partial V_2$
(by the homeo., identify $M \leftrightarrow V_1 \cup_f V_2$ & $F \leftrightarrow \partial V_i$’s)
with gluing map $f : \partial V_1 \rightarrow \partial V_2$ (homeo).

Prepare: $b$ trivial arcs $\alpha_i$ in $V_i$, $i = 1, 2$
( "trivial" := boundary parallel)
s.t. they form a trivial link $L_0$ in $M$.
We will ‘replace’ $V_2$ to get a ‘sufficiently complicate’ knot.

Setting:
$W_i := V_i - \text{(open neighborhood of } \alpha_i)$$F_0 := W_1 \cap \partial V_1 (= W_2 \cap \partial V_2)$
Theorem follows from;

Claim 1:
\[ \exists \hat{\phi} : F \to F, \text{ homeo. with } \phi := \hat{\phi}|_{F_0} \]

with

(i) \( d(D(W_1), \phi^m(D(W_2))) \to \infty \) in \( C(F_0) \) as \( m \to \infty \).
(ii) \( \hat{\phi} : F \to F \) is isotopic to \( id. \) on \( F \).

Remark : \( \phi \) acts on \( C(F_0) \) naturally.

Claim 1 \( \Rightarrow \) Thm:
Since (ii) \( \hat{\phi} \sim id. \) on \( F \),
\( \exists \Phi : \) isotopy from \( \hat{\phi} \) to \( id. \).
i.e., \( \exists \Phi : F \times [0, 1] \to F \times [0, 1] \)
s.t. \( \Phi(x, 0) = \hat{\phi}(x) \) & \( \Phi(x, 1) = x \)
Since \( \Phi|_{F \times \{1\}} = id. \), by natural extension,
we have \( h : V_2 \to V_2, \) homeo. of \( V_2 \) s.t.
\( h|_{\partial V_2} =: \hat{\phi} \) and \( h|_{F_0} =: \phi \)
Consider \( M_m := (V_1 \cup V_2)/\{f(x) = \hat{\phi}^m(y)\} \)
\( (x \in \partial V_1 \& y \in \partial V_2) \)
Since \( \hat{\phi} \sim id. \) on \( F \), we have \( M_m \cong M \) (\( \forall \ m \))
Set \( K := \alpha_1 \cup h(\alpha_2) \) in \( M_m \cong M \)
(To make a knot (not a link), we take a subsequence)
Then, for sufficiently large \( m \),
\( K \) enjoys the desired properties. \( \square \)
2. Finding $h$

Claim 3 follows from the next two claims:

**Claim 1’**: $\exists$ pseudo-Anosov map $\phi : F_0 \to F_0$
with stable/unstable lamination $\lambda^\pm \in \mathcal{PML}(F_0)$
such that
(i) $\lambda^+ \notin \overline{\mathcal{D}(W_1)}$, $\lambda^- \notin \overline{\mathcal{D}(W_2)}$, and
(ii) $\hat{\phi} : F \to F$ is isotopic to the id. on $F$ with $\hat{\phi}|_{F_0} = \phi$

**Claim 2**: For $\phi$ in Clm 1, $d(\mathcal{D}(W_1), \phi^m(\mathcal{D}(W_2))) \to \infty$ as $m \to \infty$.

**Remark**: $\mathcal{PML}(F_0)$: the space of projective measured laminations
(we omit the definition (sorry))
(note: $\mathcal{PML}(F_0) \cong S^{6g-7+2b}$)

**Key fact**: $\mathcal{D}(W_i) \hookrightarrow \mathcal{PML}(F_0)$, naturally

$\phi$: pseudo-Anosov map on $F_0$
(we also omit the definition (sorry))

**Key fact**: $\phi$ acts on $\mathcal{PML}(F_0)$ with two fixed points $\lambda^\pm$
(‘sink’ & ‘source’; like the ‘simplest’ vector field on $S^n$)
~ $\sim$ Clm2 follows from Clm 1’ (ii)
Outline of Proof of Claim 1’

By [Ichihara-Motegi, ’05],
\[\exists \phi_0: \text{p.-A. map on } F_0 \text{ with } \hat{\phi}_0 \sim id.\]
Let \(\lambda_0^\pm: \text{stable/unstable lamination of } \phi_0 \in \mathcal{PML}(F_0)\)

We modify \(\phi_0 \rightsquigarrow \phi\) s.t. stable/unstable lami. \(\not\in \mathcal{D}(W_i)\)

Subclaim: ( [Campisi-Rathbun, ’12] + \(\alpha\) )
\[\exists c: \text{simple closed curve on } F_0 \text{ s.t. } [c] \not\in \mathcal{D}(W_i)\]

Let \(\tau\): Dehn twist along \(c\)
(homeo. of \(F_0\) by cutting along \(c\), twisting, and glue back)

Set \(\phi_N := \tau^N \circ \phi_0 \circ \tau^{-1}\)
Then
\(\phi_N: \text{p.-A. with stable/unstable lamination } \tau^N(\lambda^\pm)\)
s.t. \(\hat{\phi}_N = id.\) on \(F\)

Since \(\tau^N(\lambda^\pm) \rightarrow [c] \ (N \rightarrow \infty) \& [c] \not\in \mathcal{D}(W_i)\),
for sufficiently large \(N\), \(\lambda_N^+ \not\in \mathcal{D}(W_1), \lambda_N^- \not\in \mathcal{D}(W_2)\). \(\square\)
Essentially due to [Hempel, ’01] (our proof is based on [Abrams-Schleimer, ’05])

Outline of Proof of Claim 2

Let $\phi$ in Clm 1

Suppose (for a contrary):

$d(D(W_1), \phi^m(D(W_2))) \leq Z$ ($m \to \infty$)

$\Rightarrow \exists \{\alpha^m_0, \alpha^m_1, \cdots, \alpha^m_Z\}$: curves on $F_0$

s.t. $[\alpha^m_0] \in D(W_1)$, $[\alpha^m_Z] \in (\phi)^m(D(W_2))$, $\alpha^m_k \cap \alpha^m_{k+1} = \emptyset$.

As $m \to \infty$, for each $k$, $[\alpha^m_k] \to \lambda_k$ in $\mathcal{PML}(F_0)$

$\lambda_Z = \lambda^+_N$

($\therefore \lambda^+_N \not\in \overline{D(W_2)} \Rightarrow (\phi)^m(D(W_2)) \to \lambda^+_N$)

$\lambda_{Z-1} = \lambda_Z = \lambda^+_N$

($\therefore \lambda^+_N$ is minimal & $\alpha^m_{Z-1} \cap \alpha^m_Z = \emptyset$)

$\vdots$

$\lambda_0 = \cdots = \lambda_Z = \lambda^+_N$

On the other hand, $\lambda_0 \in \overline{D(W_1)}$.

It contradicts: $\lambda^+_N \not\in \overline{D(W_1)}$. $\square$