Seifert surgeries on 
\((-2, p, p)\)-pretzel knots

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Dehn surgery on a knot

- $K$: a knot in $S^3$
- $E(K)$: the exterior of $K$ (i.e., $S^3 \setminus N^\circ(K)$)
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- $E(K)$: the exterior of $K$ (i.e., $S^3 \setminus N^\circ(K)$)

Dehn surgery: Gluing a solid torus to $E(K)$

$\gamma = [f(m)]:$ surgery slope, identified with $r \in \mathbb{Q} \cup \{1/0\}$.

$K(r)$: the manifold obtained by Dehn surgery on $K$ along $\gamma = r$. 
Exceptional surgery

Dehn surgery on a hyperbolic knot yielding a non-hyperbolic mfd.

Theorem [Thurston]

Exceptional surgeries are only finitely many for each hyperbolic knot.
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Each exceptional surgery is either:

- **Reducible surgery** (yielding a mfd. containing an essential $S^2$)
- **Toroidal surgery** (yielding a mfd. containing an essential $T^2$)
- **Seifert surgery** (yielding a Seifert manifold)

as a consequence of the Geometrization Conjecture established by Perelman ’02–’03.
Montesinos knot $M(R_1, \ldots, R_l)$

A knot admitting a diagram obtained by putting rational tangles $R_1, \ldots, R_l$ together in a circle.

$M\left(\frac{1}{2}, \frac{1}{3}, -\frac{2}{3}\right)$

- **length** of the knot $=$ minimal number of rational tangles.
- $P(a_1, \cdots, a_n) = M\left(\frac{1}{a_1}, \cdots, \frac{1}{a_n}\right): (a_1, \cdots, a_n)$-pretzel knot.
Problem

Classify all the exceptional surgeries on hyp. Montesinos knots.
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Remark [Menasco], [Oertel], [Bonahon-Siebenmann]

Non-hyperbolic Montesinos knots are

\[ T(2, n), \quad P(-2, 3, 3)(=T(3, 4)), \quad P(-2, 3, 5)(=T(3, 5)). \]

\( T(x, y) \): the \((x, y)\)-torus knot.

Remark [Moser]

Dehn surgeries on torus knots have been completely classified.
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\[ T(x, y) \]: the \((x, y)\)-torus knot.

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Dehn surgeries on torus knots have been completely classified.

Remark
Many examples of exceptional surgeries on Montesinos knots are known.
$K$: hyperbolic Montesinos knot with length $l$
Known facts: Length other than 3

\( K \): hyperbolic Montesinos knot with length \( l \)

- \( l \leq 2 \Rightarrow K \) is a two-bridge knot.
  Exceptional surgeries for them are completely classified [Brittenham-Wu ’95].

From now on, we only consider Montesinos knots with length 3.
$K$: hyperbolic Montesinos knot with length $l$

- $l \leq 2 \Rightarrow K$ is a two-bridge knot. Exceptional surgeries for them are completely classified [Brittenham-Wu ’95].

- $l \geq 4 \Rightarrow K$ admits no exceptional surgery [Wu ’96].
Known facts: Length other than 3

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- \( l \geq 4 \Rightarrow K \) admits no exceptional surgery [Wu ’96].

Remains

Exceptional surgeries on \( M(R_1, R_2, R_3) \) (i.e. \( l = 3 \))

From now on, we only consider Montesinos knots with length 3.
Known facts: Reducible / Toroidal surgery

- No reducible surgeries on Montesinos knots [Wu ’96].
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- Toroidal surgeries on Montesinos knots are completely classified [Wu ’06].
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- \(\not\exists\) reducible surgeries on Montesinos knots [Wu ’96].

- Toroidal surgeries on Montesinos knots are completely classified [Wu ’06].

- Toroidal Seifert surgeries on Montesinos knots are completely classified [Ichihara-J. ’10].
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- ∃ reducible surgeries on Montesinos knots [Wu ’96].

- Toroidal surgeries on Montesinos knots are completely classified [Wu ’06].

- Toroidal Seifert surgeries on Montesinos knots are completely classified [Ichihara-J. ’10].

Remains

Atoroidal Seifert surgeries on $M(R_1, R_2, R_3)$ (i.e. yielding a Seifert mfd. over $S^2$ with $\leq 3$ exceptional fibers)
cyclic surgery, finite surgery

Am $r$-surgery on $K$ is cyclic (resp. finite) if and only if $\pi_1(K(r))$ is cyclic (resp. finite).

Remark: Such surgeries are Seifert surgeries.
Known facts: Cyclic / Finite surgery

**cyclic surgery, finite surgery**

Am $r$-surgery on $K$ is **cyclic** (resp. **finite**)
\[
\iff \pi_1(K(r)) \text{ is cyclic (resp. finite)}. 
\]

**Remark**: Such surgeries are **Seifert** surgeries.

**Proposition [Ichihara-J.]**

**Cyclic** surgeries and **finite** surgeries on Montesinos knots are **completely classified**.
Known facts: Cyclic / Finite surgery

**cyclic surgery, finite surgery**

Am $r$-surgery on $K$ is cyclic (resp. finite) \[\Leftrightarrow \pi_1(K(r)) \text{ is cyclic (resp. finite)}.\]

**Remark:** Such surgeries are Seifert surgeries.

**Proposition [Ichihara-J.]**

Cyclic surgeries and finite surgeries on Montesinos knots are completely classified.

**Remains**

Atoroidal Seifert surgeries on $M(R_1, R_2, R_3)$ with $|\pi_1(K(r))| = \infty$ (i.e. yielding a Seifert manifold with a base orbifold $S^2(n_1, n_2, n_3)$)
Proposition [Ichihara-J.-Mizushima]

If $K = M(R_1, R_2, R_3)$ with $R_1, R_2, R_3 > 0$ (i.e. $K$ is alternating) admits an atoroidal Seifert surgery, then $K = P(a, b, c)$ with odd integers $3 \leq a < b < c$. 

An alternating hyperbolic Montesinos knot with length 3 admits no Seifert surgery.
**Known facts: atoroidal Seifert surgery**

**Proposition [Ichihara-J.-Mizushima]**

If $K = M(R_1, R_2, R_3)$ with $R_1, R_2, R_3 > 0$ (i.e. $K$ is alternating) admits an atoroidal Seifert surgery, then $K = P(a, b, c)$ with odd integers $3 \leq a < b < c$.

**Theorem [Wu ’09–’10]**

If $K = M(R_1, R_2, R_3)$ admits an atoroidal Seifert surgery, then $K = P(q_1, q_2, q_3)$ or $P(q_1, q_2, q_3, -1)$ with $(|q_1|, |q_2|, |q_3|) = (2, *, *), (3, 3, *), (3, 4, 5)$.
Known facts: atoroidal Seifert surgery

**Proposition [Ichihara-J.-Mizushima]**

If \( K = M(R_1, R_2, R_3) \) with \( R_1, R_2, R_3 > 0 \) (i.e. \( K \) is alternating) admits an atoroidal Seifert surgery, then \( K = P(a, b, c) \) with odd integers \( 3 \leq a < b < c \).

**Theorem [Wu ’09–’10]**

If \( K = M(R_1, R_2, R_3) \) admits an atoroidal Seifert surgery, then \( K = P(q_1, q_2, q_3) \) or \( P(q_1, q_2, q_3, -1) \) with \( (|q_1|, |q_2|, |q_3|) = (2, *, *), (3, 3, *), (3, 4, 5) \).

**Corollary**

An alternating hyperbolic Montesinos knot with length 3 admits no Seifert surgery.
Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then
Known facts: Summary

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(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot).

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- $K$ admits no reducible surgery,
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Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot).
\[\Rightarrow\text{ such surgeries are completely classified.}\]

(II) $l = 3$. Then

- $K$ admits no reducible surgery,
- toroidal surgeries on $K$ are classified,
Known facts: Summary

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- $K$ admits no toroidal Seifert surgery,
Suppose that a hyperbolic Montesinos knot \( K \) admits an exceptional surgery. Then

(I) \( l \leq 2 \) (i.e., \( K \) is a two-bridge knot).
\[ \Rightarrow \text{such surgeries are completely classified.} \]

(II) \( l = 3 \). Then

- \( K \) admits no reducible surgery,
- toroidal surgeries on \( K \) are classified,
- \( K \) admits no toroidal Seifert surgery,
- cyclic / finite surgeries on \( K \) are classified,
Known facts: Summary

Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot).

⇒ such surgeries are completely classified.

(II) $l = 3$. Then

- $K$ admits no reducible surgery,
- toroidal surgeries on $K$ are classified,
- $K$ admits no toroidal Seifert surgery,
- cyclic / finite surgeries on $K$ are classified,
- in addition, if $K$ is alternating, then $K$ admits no Seifert surgery.
Remains

Dehn surgeries yielding a Seifert manifold with a base orbifold $S^2(n_1, n_2, n_3)$ on non-alternating $P(q_1, q_2, q_3)$ or $P(q_1, q_2, q_3, -1)$. Here $(|q_1|, |q_2|, |q_3|) = (2, *, *)$, $(3, 3, *)$, or $(3, 4, 5)$.

Assumption: A Seifert surgery means a Dehn surgery yielding a Seifert manifold with a base orbifold $S^2(n_1, n_2, n_3)$.
Result

Remains
Dehn surgeries yielding a Seifert manifold with a base orbifold $S^2(n_1, n_2, n_3)$ on non-alternating $P(q_1, q_2, q_3)$ or $P(q_1, q_2, q_3, -1)$. Here $(|q_1|, |q_2|, |q_3|) = (2, *, *), (3, 3, *), \text{ or } (3, 4, 5)$.

Assumption: A Seifert surgery means a Dehn surgery yielding a Seifert manifold with a base orbifold $S^2(n_1, n_2, n_3)$.

Theorem [Ichihara-J.]
For odd $q \geq 1$, $P(-2, q, q)$ admits a Seifert surgery $\Leftrightarrow q = 1 \text{ or } 3$.

Remark
$P(-2, 1, 1) = 3_1 (= T(2, 3))$ and $P(-2, 3, 3) = 8_{19} (= T(3, 4))$. 
Proof of Theorem: \( r \in \mathbb{Z} \).

**Theorem [Ichihara-J.]**

For odd \( q \geq 1 \), \( P(-2, q, q) \) admits a Seifert surgery \( \iff q = 1 \) or \( 2 \).

**To show**: \( K = P(-2, q, q) \) admits no Seifert surgery if \( q \geq 5 \).

**Assume**: \( K(r) \) is Seifert for some \( q \geq 5 \) and \( r \in \mathbb{Q} \).
Proof of Theorem: $r \in \mathbb{Z}$.

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For odd $q \geq 1$, $P(-2, q, q)$ admits a Seifert surgery $\iff q = 1$ or $2$.

**To show**: $K = P(-2, q, q)$ admits no Seifert surgery if $q \geq 5$.

**Assume**: $K(r)$ is Seifert for some $q \geq 5$ and $r \in \mathbb{Q}$.

**Lemma [Wu ’09]**

For $K = M(R_1, R_2, R_3)$, if $K(r)$ is atoroidal Seifert, then $r \in \mathbb{Z}$ unless $K$ is equivalent to one of the following.

- $M(1/3, \pm 1/3, p/q)$
- $M(1/2, 1/3, p/q)$

By Lemma, we have $r \in \mathbb{Z}$. 
Proof of Theorem: \( r = 4q \pm 1 \)

Lemma [Moser], [Miyazaki-Motegi]

\( K \): hyperbolic knot with a cyclic period with period 2  
\( K' \): the factor knot of \( K \) w.r.t. the cyclic period  
\( K(r) \): Seifert mfd. with a base orbifold \( S^2(n_1, n_2, n_3) \) for \( r \in \mathbb{Z} \)  
\( K' = T_{2,q} \) with \( q \geq 3 \) \( \Rightarrow \) \( r = 4q \pm 1 \).
Proof of Theorem: Montesinos trick

\( K(4q \pm 1) \cong \text{double branched cover of } S^3 \text{ branched along } K_{q\pm}. \)
Proof of Theorem: Criterion

Fact
If $K(4q \pm 1)$ is a Seifert manifold over $S^2$, then $K_{q\pm}$ is a Montesinos knot or a torus knot.
Proof of Theorem: Criterion

Fact
If $K(4q \pm 1)$ is a Seifert manifold over $S^2$, then $K_{q\pm}$ is a Montesinos knot or a torus knot.

Lemma [Abe], [Abe-J.-Kishimoto]
$|s(K_{q\pm}) + \sigma(K_{q\pm})| \geq 4 \Rightarrow K_{q\pm}$ is not Montesinos.

sign convention
- $s(K)$: the Rasmussen invariant of $K$ with $s$(righthanded trefoil) = +2
- $\sigma(K)$: the signature of $K$ with $\sigma$(righthanded trefoil) = −2
Proof of Theorem: Rasmussen invariant of $K_{q^\pm}$

Lemma [Rasmussen]

$K$: knot  $D$: diagram of $K$  \implies s(K) \geq w(D) - O(D) + 1.$

By Lemma, we have

\[
s(K_{q^\pm}) \geq (4q - 8 + 2q - 4 \pm 1) - 4 + 1) \\
= 6q - 15 \pm 1.
\]
Proof of Theorem: signature of $K_{q\pm}$ (1)

**Lemma [H. Murakami]**

$K_0$: 2-comp. $\Rightarrow \sigma(K') - 4 \leq \sigma(K)$.

\[
\sigma(K_{q\pm}) \geq \sigma(K'_{q\pm}) - 4 \times \frac{q - 3}{2} = \sigma(K'_{q\pm}) - 2(q - 3).
\]
Proof of Theorem: signature of $K_{q^\pm}$ (2)

Lemma [Murasugi]

- $\sigma(\text{crossing}) \geq \sigma(\text{crossing}) - 2$.
- $\sigma(K) \equiv 0 \mod 4 \iff \det(K) \equiv 1 \mod 4$.
- $\sigma(K) \equiv 2 \mod 4 \iff \det(K) \equiv 3 \mod 4$.

$$\sigma(K') \geq \sigma(K') - 2(q - 3) = (-3 \mp 1) - 2q + 6 = -2q + 3 \mp 1.$$
Proof of Theorem: signature of $K_{q\pm}$ (2)

Lemma [Murasugi]

- $\sigma(\amalg) \geq \sigma(\amalg) - 2$.
- $\sigma(K) \equiv 0 \mod 4 \iff \det(K) \equiv 1 \mod 4$.
- $\sigma(K) \equiv 2 \mod 4 \iff \det(K) \equiv 3 \mod 4$.

\[
\sigma(K_{q\pm}') = \sigma(K_{q\pm}'') - 2(q - 3)
= (-3 \mp 1) - 2q + 6 = -2q + 3 \mp 1.
\]
Proof of Theorem: $K_{q\pm}$ is non-Montesinos

Recall

- $s(K_{q\pm}) \geq 6q - 15 \pm 1$.
- $\sigma(K_{q\pm}) \geq \sigma(K'_{q\pm}) - 2(q - 3)$.
- $\sigma(K'_{q\pm}) = -2q + 3 \mp 1$.
  $\Rightarrow \sigma(K_{q\pm}) \geq -4q + 9 \mp 1$.
- $q \geq 5$.

\[
|s(K_{q\pm}) + \sigma(K_{q\pm})| \geq s(K_{q\pm}) + \sigma(K_{q\pm})
\geq (6q - 15 \pm 1) + -4q + 9 \mp 1
= 2q - 6
\geq 4.
\]
Proof of Theorem: $K_{q\pm}$ is non-torus

Suppose: $K_{q\pm}$ is a torus knot.

$\Rightarrow K_{q\pm} = T(3, x)$ or $T(4, x)$ since $\text{braid}(K_{p\pm}) = 3$ or 4.
Proof of Theorem: $K_{q\pm}$ is non-torus

**Suppose**: $K_{q\pm}$ is a torus knot.

$\Rightarrow K_{q\pm} = T(3, x) \text{ or } T(4, x)$ since braid$(K_{p\pm}) = 3 \text{ or } 4$.

- $\det(K_{q\pm}) = 4q \pm 1 \geq 19$.
- $\det(T(3, x)) = 1 \text{ or } 3$. $\det(T(4, x)) = x$. 
Proof of Theorem: $K_{q\pm}$ is non-torus

**Suppose**: $K_{q\pm}$ is a torus knot.

$\Rightarrow K_{q\pm} = T(3, x)$ or $T(4, x)$ since $\text{braid}(K_{p\pm}) = 3$ or $4$.

- $\det(K_{q\pm}) = 4q \pm 1 \ (\geq 19)$.
- $\det(T(3, x)) = 1$ or $3$. $\det(T(4, x)) = x$.

$\Rightarrow K_{q\pm} = T(4, 4q \pm 1)$. 
Proof of Theorem: $K_{q\pm}$ is non-torus

\[\text{Suppose: } K_{q\pm} \text{ is a torus knot.}\]

\[\Rightarrow K_{q\pm} = T(3, x) \text{ or } T(4, x) \text{ since } \text{braid}(K_{p\pm}) = 3 \text{ or } 4.\]

\[\bullet \text{ det}(K_{q\pm}) = 4q \pm 1 \geq 19.\]

\[\bullet \text{ det}(T(3, x)) = 1 \text{ or } 3. \quad \text{det}(T(4, x)) = x.\]

\[\Rightarrow K_{q\pm} = T(4, 4q \pm 1).\]

\[s(T(4, 4q \pm 1)) = 12q + 3(\pm - 1).\]
Proof of Theorem: $K_{q^\pm}$ is non-torus

**Suppose**: $K_{q^\pm}$ is a torus knot.

$\Rightarrow K_{q^\pm} = T(3, x)$ or $T(4, x)$ since $\text{braid}(K_{p^\pm}) = 3$ or 4.

- $\det(K_{q^\pm}) = 4q \pm 1 \ (\geq 19)$.
- $\det(T(3, x)) = 1$ or $3$. \ $\det(T(4, x)) = x$.

$\Rightarrow K_{q^\pm} = T(4, 4q \pm 1)$.

$$s(T(4, 4q \pm 1)) = 12q + 3(\pm - 1).$$

$$s(K_{q^\pm}^*) \geq -(4q - 8 + 2q - 4 \mp 1) - 4 + 1$$

$$= -6q + 9 \mp 1.$$  \ $\Rightarrow s(K_{q^\pm}) \leq 6q - 9 \pm 1.$
Proof of Theorem: $K_{q\pm}$ is non-torus

\begin{proof}

Suppose $K_{q\pm}$ is a torus knot.

$\Rightarrow K_{q\pm} = T(3, x)$ or $T(4, x)$ since $\text{braid}(K_{p\pm}) = 3$ or $4$.

- $\det(K_{q\pm}) = 4q \pm 1 \geq 19$.
- $\det(T(3, x)) = 1$ or $3$.  $\det(T(4, x)) = x$.

$\Rightarrow K_{q\pm} = T(4, 4q \pm 1)$.

\[
\begin{align*}
\det(T(4, 4q \pm 1)) &= 12q + 3(\pm - 1).
\end{align*}
\]

\[
\begin{align*}
\det(K_{q\pm}^*) &\geq -(4q - 8 + 2q - 4 \mp 1) - 4 + 1 \\
&= -6q + 9 \mp 1.
\end{align*}
\]

$\Rightarrow \det(K_{q\pm}) \leq 6q - 9 \pm 1$.

$\Rightarrow \det(T(4, 4q \pm 1)) > \det(K_{q\pm})$ a contradiction.
\end{proof}