Exceptional surgeries on $(−2, p, p)$-pretzel knots

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Dehn surgery on a knot

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- $E(K)$: the exterior of $K$ (i.e., $S^3 \setminus N^\circ(K)$)
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- \( E(K) \): the exterior of \( K \) (i.e., \( S^3 \setminus N^\circ(K) \))

**Dehn surgery:** Gluing a solid torus to \( E(K) \)

\[ \gamma = \left[ f(m) \right] \]: surgery slope, identified with \( r \in \mathbb{Q} \cup \{1/0\} \).

\( K(r) \): the manifold obtained by Dehn surgery on \( K \) along \( \gamma = r \).
Types of Dehn surgeries

As a consequence of the geometrization conjecture, a Dehn surgery on a knot is one of the following:

- **Hyperbolic** surgery (yielding a hyperbolic mfd.)
- **Seifert** surgery (yielding a Seifert fibered mfd.)
- **Toroidal** surgery (yielding a mfd. containing an essential $T^2$)
- **Reducible** surgery (yielding a mfd. containing an essential $S^2$)
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**Problem**

For a given knot $K$ and $r \in \mathbb{Q}$, determine a type of $K(r)$. 
Pretzel knot $P(a_1, \ldots, a_n)$

A knot admitting a diagram obtained by putting $n$ tangles consisting of $a_i$-half twists ($i = 1, \ldots, n$) together in a circle.

$P(-2, 3, 3)$
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Today we study Dehn surgeries on $P(-2, p, q)$ with odd positive integers $p, q$. 
Hyperbolicity of $P(-2, p, q)$

Fact [Menasco], [Oertel], [Bonahon-Siebenmann]
Non-hyperbolic pretzel knots are

- $P(-2, 1, q) = T(2, q + 2)$,
- $P(-2, 3, 3) = T(3, 4)$, and
- $P(-2, 3, 5) = T(3, 5)$.

$T(x, y)$: the $(x, y)$-torus knot.
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Dehn surgeries on torus knots have been completely classified.
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Dehn surgeries on torus knots have been completely classified.

Hereafter we focus on hyperbolic $P(-2, p, q)$. 
Reducible surgeries on $P(-2, p, q)$

Recall: A Dehn surgery on a knot is either Hyperbolic, Seifert, Toroidal, or Reducible.

Proposition [Wu]
A hyperbolic $P(-2, p, q)$ admits no reducible surgery.
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**Conjecture (Cabling conjecture)**

A hyperbolic knot admits no reducible surgery.
Result 1: **Seifert surgeries on** $P(-2, p, q)$

**Fact** [Ichihara-J.], [Futer-Ishikawa-Kabaya-Mattman-Shimokawa]

$K = P(-2, p, q)$ admits **Seifert** surgery with $|\pi_1(K(r))| < \infty$

$\iff K = P(-2, 3, 7) \text{ or } P(-2, 3, 9)$. 

Recall: $P(-2, 1, 1) = T(2, 3)$ and $P(-2, 3, 3) = T(3, 4)$. 

Open problem.

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**Theorem A** [Ichihara-J.-Kabaya]

For odd $p \geq 1$, $P(-2, p, p)$ admits a **Seifert** surgery $\Leftrightarrow p = 1$ or 3.
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**Corollary**

A hyperbolic $P(-2, p, p)$ admits **no Seifert** surgery.
Result 1: \textbf{Seifert surgeries on }$P(-2, p, q)$

\textbf{Fact [Ichihara-J.], [Futer-Ishikawa-Kabaya-Mattman-Shimokawa]}

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\textbf{Theorem A [Ichihara-J.-Kabaya]}

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\textbf{Corollary}

A hyperbolic $P(-2, p, p)$ admits \textbf{no Seifert} surgery.

\textbf{Open problem}

Determine Seifert surgeries on $P(-2, p, q)$ with $3 \leq p < q$. 
Outline of the proof of Theorem A

Applying the Montesinos trick, we obtain a link $L_r \subset S^3$ such that the double branched covering of $S^3$ branched along $L_r$ is $P(-2,p,p)(r)$ (Remark: $P(-2,p,p)$ is strongly invertible).
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$\Rightarrow$ If $L_r$ is neither Montesinos nor Seifert for $\forall r \in \mathbb{Q}$,

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If \( P(-2, p, p)(r) \) is a Seifert manifold,
then \( L_r \) is a Montesinos link or a Seifert link.

\[ \Rightarrow \] If \( L_r \) is neither Montesinos nor Seifert for \( \forall r \in \mathbb{Q} \),
then \( P(-2, p, p) \) admits no Seifert surgery.

To show that \( L_r \) is not a Montesinos knot, we use the following.

\( s(K) \): the Rasmussen invariant of a knot \( K \)
\( \sigma(K) \): the signature of a knot \( K \)

**Lemma**

\[ |s(L_r) - \sigma(L_r)| \geq 4 \Rightarrow L_r \text{ is not a Montesinos knot.} \]
**Proposition [Wu]**

$P(-2, p, q)(r)$ is toroidal $\iff r = 2p + 2q$. 

**Question.**

Is $N_{p,q}$ Seifert or hyperbolic?
Toroidal surgery on $P(-2, p, q)$

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\[ \Rightarrow P(-2, p, q)(2p + 2q) \text{ contains a unique incompressible torus } T. \]

\[ \Rightarrow \text{Cutting } P(-2, p, q)(2p + q) \text{ along } T, \text{ we have two components.} \]

One of them is a twisted \( I \)-bundle over the Klein bottle.
**Toroidal surgery on** $P(-2, p, q)$

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$\Rightarrow P(-2, p, q)(2p + 2q)$ contains a unique incompressible torus $T$.

$\Rightarrow$ Cutting $P(-2, p, q)(2p + q)$ along $T$, we have two components.

One of them is a twisted $I$-bundle over the Klein bottle.

$N_{p,q}$: the other component

**Question**

Is which $N_{p,q}$ Seifert or hyperbolic?
Result 2: JSJ pieces of $P(-2, p, q)(2p + 2q)$

$C$: the 3-component chain-link

**Theorem B** [Ichihara-J.-Kabaya]

$N_{p,q} = C \left( \frac{1 + p}{1 - p}, \frac{1 + q}{1 - q} \right)$. 
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Theorem B [Ichihara-J.-Kabaya]

$$N_{p,q} = C \left( \frac{1+p}{1-p}, \frac{1+q}{1-q} \right).$$

Dehn surgeries on $C$ are completely classified [Martelli-Petronio].

Corollary

- $N_{3,q}$ is the Seifert manifold $(D^2; (3, 1), (\frac{q-3}{2}, \frac{q-1}{2}))$.
- The others are hyperbolic.
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In particular, $N_{5,5}$ is the “figure-eight knot sister manifold”.