Exceptional surgeries on Montesinos knots

Kazuhiro Ichihara
Nihon University
College of Humanities and Sciences

Based on joint works with

In Dae Jong  (Osaka Prefecture University)
Yuichi Kabaya  (Osaka University)
Hidetoshi Masai  (Tokyo Institute of Technology)

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1. Introduction
Dehn surgery on a knot

- $K$: a knot in the 3-sphere $S^3$
- $E(K)$: the exterior of $K$ ($= S^3 - (\text{open nbd. of } K)$)

**Dehn surgery on $K$**

Gluing a solid torus $V$ to $E(K)$ to obtain a closed manifold.
Dehn surgery on a knot

- $K$: a knot in the 3-sphere $S^3$
- $E(K)$: the exterior of $K$ (= $S^3$–(open nbd. of $K$))

Dehn surgery on $K$

Gluing a solid torus $V$ to $E(K)$ to obtain a closed manifold.

This gives a BRIDGE between Knot Theory & 3-mfd Theory

Theorem [Lickorish (1962), Wallace (1960)]

Every closed orientable 3-manifold is obtained by Dehn surgery on a link in $S^3$. 
Dehn surgery on a knot

- \( K \): a knot in the 3-sphere \( S^3 \)
- \( E(K) \): the exterior of \( K \) \((= S^3-(\text{open nbd. of } K))\)

Dehn surgery on \( K \)

Gluing a solid torus \( V \) to \( E(K) \) to obtain a closed manifold.

\[
\text{Notation}
\]
For \( f : \partial V \to \partial E(K) \) and \( m \): meridian of \( V \),
\[
r = \lfloor f(m) \rfloor : \text{ surgery slope }, \text{ regard as } r \in \mathbb{Q} \cup \{1/0\}.
\]

\( K(r) \): the manifold obtained by surgery on \( K \) along \( r \).
Motivation

Theorem [Thurston (1978)]

Dehn surgeries on a hyperbolic knot
(i.e., knot with hyperbolic complement)
yielding a non-hyperbolic manifold are only finitely many.
Motivation

Theorem [Thurston (1978)]

Dehn surgeries on a hyperbolic knot (i.e., knot with hyperbolic complement) yielding a non-hyperbolic manifold are only finitely many.

Exceptional surgery

Dehn surgery on a hyperbolic knot yielding a non-hyperbolic manifold is called exceptional surgery.
Exceptional surgeries on Montesinos knots

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Motivation

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Dehn surgeries on a hyperbolic knot
(i.e., knot with hyperbolic complement)
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Exceptional surgery

Dehn surgery on a hyperbolic knot yielding
a non-hyperbolic manifold is called exceptional surgery.

An exceptional surgery is either:

- **Reducible** surgery (yielding a mfd. containing an essential \( S^2 \))
- **Toroidal** surgery (yielding a mfd. containing an essential \( T^2 \))
- **Seifert** surgery (yielding a Seifert fibered manifold) as a consequence of the Geometrization Conjecture established by Perelman (2002-03).
Montesinos knot $M(R_1, \ldots, R_l)$

A knot admitting a diagram obtained by putting rational tangles $R_1, \ldots, R_l$ together.

arcs on a 4-punctured sphere, and $\frac{1}{2}$-tangle

length of the knot
$= \text{minimal number of rational tangles}$

$M(\frac{1}{2}, \frac{1}{3}, -\frac{2}{3})$

$P(a_1, \cdots, a_n) = M(\frac{1}{a_1}, \cdots, \frac{1}{a_n}) : (a_1, \cdots, a_n)$-pretzel knot.
Problem

Classify all the exceptional surgeries on hyperbolic Montesinos knots.

Remark [Menasco], [Oertel], [Bonahon-Siebenmann]

Non-hyperbolic Montesinos knots are $T(2; n)$, $P(-2; 3; 3)$ ($= T(3; 4)$), $P(-2; 3; 5)$ ($= T(3; 5)$).
Problem

Classify all the exceptional surgeries on hyperbolic Montesinos knots.

Remark [Menasco], [Oertel], [Bonahon-Siebenmann]

Non-hyperbolic Montesinos knots are

\[ T(2, n), \quad P(-2, 3, 3) (= T(3, 4)), \quad P(-2, 3, 5) (= T(3, 5)). \]

\[ T(x, y) : \text{the } (x, y)-\text{torus knot.} \]
2. Known facts
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Length other than 3 & Reducible / Toroidal cases

\( K \) : hyperbolic Montesinos knot with length \( l \)
Length other than 3 & Reducible / Toroidal cases

\( K \) : hyperbolic Montesinos knot with length \( l \)

- \( l \leq 2 \implies K \) is a two-bridge knot. Exceptional surgeries for them are completely classified [Brittenham-Wu (1995)].
Length other than 3 & Reducible / Toroidal cases

\( K \) : hyperbolic Montesinos knot with length \( l \)

- \( l \leq 2 \Rightarrow K \) is a two-bridge knot. Exceptional surgeries for them are completely classified [Brittenham-Wu (1995)].

- \( l \geq 4 \Rightarrow K \) admits no exceptional surgery [Wu (1996)].
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\[ l \geq 4 \Rightarrow K \text{ admits no exceptional surgery [Wu (1996)].} \]

\[ \exists \text{ reducible surgeries on Montesinos knots [Wu (1996)].} \]

\[ K : \text{ hyperbolic Montesinos knot with length } l \]

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Known facts

\[ l \leq 2 \Rightarrow K \text{ is a two-bridge knot.} \]

Exceptional surgeries for them are completely classified [Brittenham-Wu (1995)].

\[ l \geq 4 \Rightarrow K \text{ admits no exceptional surgery [Wu (1996)].} \]

\[ l > 4 \Rightarrow \exists \text{ reducible surgeries on Montesinos knots [Wu (1996)].} \]

\[ \text{Toroidal surgeries on Montesinos knots are completely classified [Wu (2006)].} \]

Remains

Seifert surgeries on \(M(R_1, R_2, R_3)\)
Known facts: Atoroidal Seifert surgery

The following are shown by [Wu (2009,2010)].

If a Montesinos knot $K$ of length 3 admits an atoroidal Seifert fibered surgery, then $K$ is equivalent to

\[ M(1; q_1; 1q_2; 1q_3) = P(q_1; q_2; q_3) \]
\[ M(1q_1; 1q_2; 1q_3; -1) = P(q_1; q_2; q_3; -1) \] with $q_i > 0$

In either case, up to relabeling, $(|q_1|; |q_2|; |q_3|) = (2; |q_2|; |q_3|)$; $(3; 3; |q_3|)$; or $(3; 4; 5)$.

$\text{I}(\text{non-pretzel case})$

\[ M(-2=3; 1=3; 2=5) \]
\[ M(-1=2; 1=q; 2=5) \] for some $q \geq 3$ odd.
Known facts: Atoroidal Seifert surgery

The following are shown by [Wu (2009,2010)].

If a Montesinos knot $K$ of length 3 admits an atoroidal Seifert fibered surgery, then $K$ is equivalent to

- **(pretzel case)**
  1. $M(\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}) = P(q_1, q_2, q_3)$
  2. $M(\frac{1}{q_1}, \frac{1}{q_2}, \frac{1}{q_3}, -1) = P(q_1, q_2, q_3, -1)$ with $q_i > 0$

In either case, up to relabeling,

$$(|q_1|, |q_2|, |q_3|) = (2, |q_2|, |q_3|), (3, 3, |q_3|), \text{ or } (3, 4, 5).$$

- **(non-pretzel case)**
  1. $M(-2/3, 1/3, 2/5)$
  2. $M(-1/2, 1/3, 2/(2a + 1))$ and $a \in \{3, 4, 5, 6\}$
  3. $M(-1/2, 1/q, 2/5)$ for some $q \geq 3$ odd.
3. Toroidal Seifert surgery
Known facts: Toroidal Seifert surgery

Recall: Each exceptional surgery is either:

- Reducible,
- Toroidal,
- Seifert.
Known facts: Toroidal Seifert surgery

Recall: Each exceptional surgery is either:

- Reducible (conjectured: $\not\exists$ (Cabling Conjecture)),
- Toroidal,
- Seifert.
Recall: Each exceptional surgery is either:

- Reducible \((\text{conjectured: }\exists \text{ (Cabling Conjecture)})\),
- Toroidal,
- Seifert.

Remark [Eudave-Muñoz (2002)]

They are not exclusive (i.e., \(\exists\) non-empty intersection).
**Known facts: Toroidal Seifert surgery**

<table>
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<tr>
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**Remark [Eudave-Muñoz (2002)]**

They are **not** exclusive (i.e., $\exists$ non-empty intersection).

**Theorem [Motegi (2003)]**

A hyperbolic knot $K$ with $|\text{Sym}^*(K)| > 2$

has **no toroidal Seifert surgery**.
Known facts: Toroidal Seifert surgery

Recall: Each exceptional surgery is either:

- Reducible (conjectured: $\exists$ (Cabling Conjecture)),
- Toroidal,
- Seifert.

Remark [Eudave-Muñoz (2002)]
They are not exclusive (i.e., $\exists$ non-empty intersection).

Theorem [Motegi (2003)]
A hyperbolic knot $K$ with $|\text{Sym}^*(K)| > 2$
has no toroidal Seifert surgery.
In particular, other than the trefoil knot, no two-bridge knots admit toroidal Seifert surgeries.
Results: Toroidal Seifert surgery

Theorem [I.-Jong (2010)]

Montesinos knots admit no toroidal Seifert surgeries other than the trefoil knot.

Corollary

No hyperbolic Montesinos knots have toroidal Seifert surgery.
Results: Toroidal Seifert surgery

Theorem [I.-Jong (2010)]

Montesinos knots admit no toroidal Seifert surgeries other than the trefoil knot.

Corollary

No hyperbolic Montesinos knots have toroidal Seifert surgery.

Key Proposition [I.-Motegi-Song (2008)]

If a small hyperbolic knot $K$ in $S^3$ admits a toroidal Seifert fibered, then $K$ is fibered and the surgery is longitudinal.
4. Cyclic/Finite surgery
Cyclic / Finite surgery

As a consequence of the Geometrization Conjecture established by Perelman (2002-03), 3-manifolds with cyclic or finite fundamental groups are all Seifert fibered.
Cyclic / Finite surgery

As a consequence of the Geometrization Conjecture established by Perelman (2002-03), 3-manifolds with cyclic or finite fundamental groups are all Seifert fibered.

Problem

On (hyperbolic) knots in $S^3$, determine all Dehn surgeries giving 3-manifolds with cyclic or finite fundamental groups.

We call such surgeries cyclic surgeries / finite surgeries respectively.
Theorem

We give a complete classification of cyclic / finite surgeries on Montesinos knots.

Theorem [I.-Jong (2009)]

\(K: \) hyperbolic Montesinos knot

\(K(r): \) the manifold obtained by surgery on \(K\) along \(r\).

1. If \(1(K(r))\) is cyclic, then \(K = P(-2;3;7)\) and \(r = 18\) or \(19\).
2. If \(1(K(r))\) is acyclic finite, then \(K = P(-2;3;7)\) and \(r = 17\), or \(K = P(-2;3;9)\) and \(r = 22\) or \(23\).
Theorem

We give a complete classification of cyclic / finite surgeries on Montesinos knots.

**Theorem [I.-Jong (2009)]**

\( K \) : hyperbolic Montesinos knot

\( K(r) \) : the manifold obtained by surgery on \( K \) along \( r \).
We give a complete classification of cyclic / finite surgeries on Montesinos knots.

**Theorem [I.-Jong (2009)]**

\[ K : \text{hyperbolic Montesinos knot} \]
\[ K(r) : \text{the manifold obtained by surgery on } K \text{ along } r. \]

- If \( \pi_1(K(r)) \) is cyclic,
  then \( K = P(-2, 3, 7) \) and \( r = 18 \) or \( 19 \).
We give a complete classification of cyclic / finite surgeries on Montesinos knots.

Theorem [I.-Jong (2009)]

Let $K$ be a hyperbolic Montesinos knot, and $K(r)$ be the manifold obtained by surgery on $K$ along $r$.

1. If $\pi_1(K(r))$ is cyclic, then $K = P(-2,3,7)$ and $r = 18$ or 19.
2. If $\pi_1(K(r))$ is acyclic finite, then $K = P(-2,3,7)$ and $r = 17$, or $K = P(-2,3,9)$ and $r = 22$ or 23.
Key ingredients

Use Heegaard Floer Homology

Fact [Ozsvath-Szabo, 2005]

\[ \pi_1(M) \text{ is cyclic} / \text{ finite} \Rightarrow M \text{ is an L-space} \]
Key ingredients

Use Heegaard Floer Homology

Fact [Ozsvath-Szabo, 2005]

\[ \pi_1(M) \text{ is cyclic / finite } \Rightarrow M \text{ is an L-space} \]

- (c.f. [Ozsváth-Szabó, 2005])
  If a knot \( K \) in \( S^3 \) admits an L-space surgery, then every non-zero coeff. of the Alexander polynomial is \( \pm 1 \).

- ([Ni, 2007])
  If a knot in \( S^3 \) admits an L-space surgery, then it must be a fibered knot.
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5. Seifert surgery
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Theorem [I.-Jong, 2011(preprint)]

Any hyperbolic pretzel knot $P(p, q, q)$ with $p, q \geq 2$ admits no Seifert fibered surgery.

Theorem [I.-Jong-Kabaya, 2012]

Any hyperbolic pretzel knot $P(-2, q, q)$ with $q \geq 5$ admits no Seifert fibered surgery.
Results

**Theorem [I.-Jong, 2011(preprint)]**

Any hyperbolic pretzel knot $P(p, q, q)$ with $p, q \geq 2$ admits no Seifert fibered surgery.

**Theorem [I.-Jong-Kabaya, 2012]**

Any hyperbolic pretzel knot $P(-2, q, q)$ with $q \geq 5$ admits no Seifert fibered surgery.

c.f. Toroidal surgery on $P(p, q, q)$ [I.-J.-K.]

Let $K = P(-2, p, q)$ with odd integers $p, q \geq 3$. Suppose that $K(r)$ admits a non-trivial JSJ-decomposition. Then the pieces of the decomp. of $K(r)$ are a twisted I-bundle over the Klein bottle and the manifold given by $(\frac{1+p}{1-p}, \frac{1+q}{1-q})$-surgery on the 3-chain-link.
Key ingredients

$P(p, q, q)$ is strongly invertible.
Key ingredients

\[ P(p, q, q) \text{ is strongly invertible.} \]

Montesinos trick [Montesinos, 1975]

For a strongly invertible knot \( K \) in \( S^3 \) and a slope \( r \), there exists a link \( L_r \) in \( S^3 \) such that \( K(r) \) is homeomorphic to the double-branched cover \( \Sigma_{L_r} \) of \( S^3 \) along \( L_r \).
Key ingredients

\[ P(p, q, q) \text{ is strongly invertible.} \]

Montesinos trick [Montesinos, 1975]

For a strongly invertible knot \( K \) in \( S^3 \) and a slope \( r \), there exists a link \( L_r \) in \( S^3 \) such that \( K(r) \) is homeomorphic to the double-branched cover \( \Sigma_{L_r} \) of \( S^3 \) along \( L_r \).

Main case

The link \( L_r \) s.t. \( K(r) \cong \Sigma_{L_r} \) is a Montesinos knot.
Task

Show that the given knot is not a Montesinos knot.

Criterion

Let $s(K)$ be the Rasmussen invariant, and $\sigma(K)$ the signature of a knot $K$. Then $|s(K) - \sigma(K)| \geq 4 \Rightarrow K$ is not a Montesinos knot.
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Criterion

Task
Show that the given knot is not a Montesinos knot.

Use Rasmussen invariant.

Key Lemma
Let $s(K)$ be the Rasmussen invariant, and $\sigma(K)$ the signature of a knot $K$. Then $|s(K) - \sigma(K)| \geq 4 \Rightarrow K$ is not a Montesinos knot.
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Remains
If a Montesinos knot $K$ of length 3 admits a Seifert fibered surgery, then $K$ is equivalent to

- **(pretzel case)**
  1. $P(\pm 2, a, b)$ with $a \neq b$
  2. $P(3, 3, c)$ with $c \leq -3$
  3. $P(3, 3, d, -1)$ with $d \geq 3$
  4. $P(3, -3, \pm e)$ with $e \geq 4$
  5. $P(3, \pm 4, \pm 5)$, $P(3, \pm 4, \mp 5)$, $P(3, 4, 5, -1)$

- **(non-pretzel case)**
  6. $M(-2/3, 1/3, 2/5)$
  7. $M(-1/2, 1/3, 2/(2p + 1))$ and $p \in \{3, 4, 5, 6\}$
  8. $M(-1/2, 1/q, 2/5)$ for some $q \geq 3$ odd.
The pretzel knot $P(3, 4, 5)$ admits no Seifert fibered surgery.
“Theorem” [I.-Masai, in preparation]

The pretzel knot $P(3, 4, 5)$ admits no Seifert fibered surgery.

This is verified by using computer-aided procedure developed in

B. Martelli, C. Petronio, F. Roukema
Exceptional Dehn surgery on
the minimally twisted five-chain link
preprint, arXiv:1109.0903

A program downloadable from
http://www.dm.unipi.it/~martelli/research.html
**Procedure**

The procedure depends upon

- **SnapPy** (based on **SnapPea**): computer software calculates various hyperbolic invariants for 3-manifolds.

- **Moser’s algorithm**
  
  Harriet H. Moser
  Proving a manifold to be hyperbolic once it has been approximated to be so
  Algebraic & Geometric Topology 9 (2009) 103–133.

  may rigorously guarantee the calculations by SnapPy.
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  may rigorously guarantee the calculations by SnapPy.

It can give us a complete classification of exceptional surgeries on a given hyperbolic link if we are lucky.

We are now finding more possible applications...