Exceptional surgeries on alternating knots

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Table of contents

Introduction
    Dehn surgery
    Exceptional surgery
    Result

Outline of Proof
    $t(d) \geq 9$
    arborescent knots
    Remaining cases
    Using Computer

Procedure of Computation (rough sketch)
    FEF.py
Dehn surgery on a knot

\( K \): a knot in the 3-sphere \( S^3 \)

**Dehn surgery on** \( K \)

An operation to create a (new) 3-manifold from \( K \).

This gives a *fundamental operation* to study 3-manifolds.
Exceptional surgeries on alternating knots

K. Ichihara

Introduction

Dehn surgery

Exceptional surgery

Result

Outline of Proof

$t(d) \geq 9$

arborescent knots

Remaining cases

Using Computer

Procedure of Computation (rough sketch)

FEF.py

Motivation

Hyperbolic Dehn Surgery Theorem [Thurston (1978)]

Only finitely many Dehn surgeries on a hyperbolic knot (i.e., knot with hyperbolic complement) yield non-hyperbolic manifolds.

Exceptional surgery

A Dehn surgery on a hyperbolic knot is called exceptional if it yields a non-hyperbolic manifold.

Ultimate Goal

Classify all the exceptional surgeries on hyperbolic knots in $S^3$. 
Result

Theorem [I.-Masai, in preparation]

Let $K$ be a hyperbolic alternating knot in $S^3$. If $K$ admits an exceptional surgery, then $K$ is a 2-bridge knot or a pretzel knot of length 3.

Facts

Complete classifications of exceptional surgeries have been known for

- 2-bridge knots [Brittenham-Wu, 2001]
Table of contents

Introduction
  Dehn surgery
  Exceptional surgery
  Result

Outline of Proof
  \( t(d) \geq 9 \)
  arborescent knots
  Remaining cases
  Using Computer

Procedure of Computation (rough sketch)
  FEF.py
Upper Bound of Twist Number

Let $K$ be a hyperbolic alternating knot in $S^3$.

Fact [Lackenby (2000)]

If $K$ has a prime alternating diagram $D$ satisfying $t(D) \geq 9$, then $K$ admits no exceptional surgeries.

Here $t(D)$ denotes the number of twists.

A twist is defined as either;
- a maximal connected collection of bigon regions in $D$
- or an isolated crossing adjacent to no bigon regions.
All alternating knots with $t(D) \leq 5$ are arborescent.

Most alternating knots with $t(D) \leq 8$ are arborescent.

A knot $K$ is called an arborescent knot if it can be obtained by summing and gluing several rational tangles together.

Fact (Wu, etc.)

All the exceptional surgeries on hyperbolic alternating arborescent knots are completely classified.
Remaining cases

Lemma

Let $K$ be a hyperbolic non-arborescent alternating knot with a prime alternating diagram $D$ satisfying $t(D) \leq 8$.

Then $K$ is obtained by adding crossings to twists from

- the Borromean rings $6_2^3$,
- the link with at most 2 twist regions add to $6_2^3$,
- or the knot $8_{18}$.

Actually, we can show that there are 9 templates.

Thus, for example, $K$ can be obtained by twisting along augmented loops from the following links...
Exceptional surgeries on alternating knots

K. Ichihara

Introduction

Dehn surgery

Exceptional surgery

Result

Outline of Proof

t(d) 9

arborescent knots

Remaining cases

Using Computer

Procedure of Computation (rough sketch)

FEF.py
Remaining cases

It suffice to get a complete classification of exceptional surgeries on the links like those as illustrated.

The number of such links are at most

$$4^6 + 4^7 + 7 \times 4^8 = 126976$$

By using symmetry, and other restrictions, we reduce the number into 29017.

Target: Exceptional surgeries on these 29017 links.

We listed up these links by using computer program.
Using Computer

To study exceptional surgeries on the links, we further used a computer program developed in;

B. Martelli, C. Petronio, F. Roukema

**Exceptional Dehn surgery on the minimally twisted five-chain link**

preprint, arXiv:1109.0903

The codes are downloadable at

http://www.dm.unipi.it/~martelli/Dehn.html
Ingredients

The program relies upon

- **SnapPy** (based on SnapPea): computer software calculates various hyperbolic invariants for 3-manifolds.

To get a rigorous proof, it needs to apply **Moser’s algorithm**.

Harriet H. Moser

**Proving a manifold to be hyperbolic once it has been approximated to be so**

Algebraic & Geometric Topology 9 (2009) 103–133.

It can possibly give us a complete classification of exceptional surgeries on a given hyperbolic link.
Computation time

- We have 29017 links to investigate.
- For ONE link, we need around 8 HOURS by PC.

⇒ We need to reduce the computation time.

Fact [Wu] + Observation

Suppose
- the edges go through a crossing circle are anti-parallel
- after ”smoothing”, #(knot components) > 1.

Then the links obtained by twisting more than once along augmented loops have NO exceptional surgeries.

⇒ We can reduce the computation time to about 1/10.

Still (by very rough estimate) we need more than 20000 hours by PC...
Exceptional surgeries on alternating knots

K. Ichihara

Introduction

Dehn surgery

Exceptional surgery

Result

Outline of Proof

\[ t(d) \geq 9 \]

Arborescent knots

Remaining cases

Using Computer

Procedure of Computation (rough sketch)

FEF.py

TSUBAME

- TSUBAME is the supercomputer of Tokyo Tech.
- Intuitively we can use many machines simultaneously.
- I ”rent” 640 machines.
Table of contents

Introduction
  Dehn surgery
  Exceptional surgery
  Result

Outline of Proof
  \( t(d) \geq 9 \)
  arborescent knots
  Remaining cases
  Using Computer

Procedure of Computation (rough sketch)
  FEF.py
Martelli’s Code

```
find_exceptional_filling.py:
```

Given cusped hyperbolic 3-manifold, find geometric solution for consistency equations, output data for the Moser’s test, and find the slopes of length < 6.1 on the maximal cusp.

Recall that:

**Fact [Agol, Lackenby (2000)]**

Any Dehn surgery on a hyperbolic knot along a slope of length > 6 is not exceptional, i.e., gives a hyperbolic 3-manifold.
Exceptional surgeries on alternating knots

K. Ichihara

Introduction
Dehn surgery
Exceptional surgery
Result

Outline of Proof
\( t(d) \geq 9 \)
arborescent knots
Remaining cases
Using Computer

Procedure of Computation (rough sketch)
FEF.py

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**Procedure**

Input: a triangulation of cusped hyperbolic 3-manifold

1. Canonize the given triangulation.
   Does the obtained triangulation contain flat or nearly flat tetrahedra?
   - \( Y \rightarrow (3) \) [ N. \( \rightarrow (2) \) ]

2. Randomize the triangulation.
   - \( \rightarrow (1) \)
Exceptional surgeries on alternating knots

K. Ichihara

Introduction
Dehn surgery
Exceptional surgery
Result

Outline of Proof
\( t(d) \geq 9 \)
arborescent knots
Remaining cases
Using Computer

Procedure

(3)
Try to find a nicely approximated hyperbolic structure.
Can such a solution be found?

\[ \text{Y. } \rightarrow (4) \quad \text{N. } \rightarrow (2) \]

(4)
Output data for the Moser’s test.
Find and fix a nearly maximal cusp by horoball expansions.
List up the slopes of length \(< 6.1\) on the cusp.
Do such slopes exist?

\[ \text{Y. } \rightarrow (5) \quad \text{N. } \rightarrow \text{(end)} \]
(5) Perform a Dehn filling along a slope in the list.  
Try to find a nicely approximated hyperbolic structure.  
Can such a solution be found?  

\[ \text{Y. } \rightarrow \text{(7) } \quad \text{N. } \rightarrow \text{(6) } \]

(6) Randomize the triangulation.  

\[ \rightarrow \text{(5) } \]
Procedure

(7)

Output data for the Moser’s test.

Recursively apply this procedure for the obtained cusped hyperbolic 3-manifolds.

After applying `find_exceptional_filling.py`, we used the Moser’s test for all the output data.

Actually, applying this procedure, we treated more than 7 million hyperbolic 3-manifolds...

Finally, we can conclude there are no exceptional surgeries on hyperbolic non-arborescent alternating knots with a prime alternating diagram $D$ satisfying $t(D) \leq 8$. 
Thank you for your attention!

감사합니다

ありがとうございます。

谢谢