On Seifert fibered surgeries on knots

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Dehn surgery on a knot

- \( K \): a knot in \( S^3 \)
- \( E(K) \): the exterior of \( K \) (i.e., \( S^3 \setminus N^\circ(K) \))
Dehn surgery on a knot

- $K$: a knot in $S^3$
- $E(K)$: the exterior of $K$ (i.e., $S^3 \setminus N^\circ(K)$)

Dehn surgery: Gluing a solid torus to $E(K)$

$\gamma = [f(m)]$: surgery slope, identified with $r \in \mathbb{Q} \cup \{1/0\}$.

$K(r)$: the manifold obtained by Dehn surgery on $K$ along $\gamma = r$. 
By the Geometrization due to Perelman, a Dehn surgery on a knot is one of the following:

- **Hyperbolic surgery** (yielding a hyperbolic mfd.)
- **Seifert surgery** (yielding a Seifert mfd.)
- **Toroidal surgery** (yielding a mfd. containing an essential $T^2$)
- **Reducible surgery** (yielding a mfd. containing an essential $S^2$)
Types of Dehn surgeries

By the Geometrization due to Perelman, a Dehn surgery on a knot is one of the following:

- **Hyperbolic** surgery (yielding a hyperbolic mfd.)
- **Seifert** surgery (yielding a Seifert mfd.)
- **Toroidal** surgery (yielding a mfd. containing an essential $T^2$)
- **Reducible** surgery (yielding a mfd. containing an essential $S^2$)

Cabling Conjecture [González-Acuña and Short]
All reducible surgeries have already completely classified.

Remark [Eudave-Muñoz] [Gordon and Luecke]
The classification is not exclusive.
(∃ infinitely many hyperbolic knots each of which admits a toroidal Seifert surgery.)
Theorem [Motegi ’02]

\( K \) : 2-bridge knot, \( K(r) \) : toroidal Seifert manifold
\( \Rightarrow \) \( K = T(2, 3) \) and \( r = 0 \).
Toroidal Seifert surgery

Theorem [Motegi ’02]

\( K : \) 2-bridge knot, \( K(r) : \) toroidal Seifert manifold
\( \Rightarrow K = T(2, 3) \) and \( r = 0. \)

Theorem 1 [Ichihara-J. ’10]

\( K : \) Montesinos knot, \( K(r) : \) toroidal Seifert manifold
\( \Rightarrow K \) is the trefoil knot and \( r = 0. \)

Theorem 2 [Ichihara-J. ’13(’09)]

\( K : \) alternating knot, \( K(r) : \) toroidal Seifert manifold
\( \Rightarrow \) • \( K = T(2, \pm 3) \) and \( r = 0, \) or
  • \( K = T(2, p) \# T(2, q) \) and \( r = 2(p + q) \) with \( |p|, |q| \geq 3. \)
(small) **Seifert surgery**

**Theorem** [Brittenham-Wu ’01]
All **Seifert** surgeries on **2-bridge** knots are completely classified.
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All Seifert surgeries on 2-bridge knots are completely classified.

Theorem 3 [Ichihara-J. ’09]
All Seifert surgeries on Montesinos knots with $|\pi_1 K(r)| < \infty$ are completely classified.
(small) **Seifert surgery**

**Theorem** [Brittenham-Wu ’01]
All Seifert surgeries on 2-bridge knots are completely classified.

**Theorem 3** [Ichihara-J. ’09]
All Seifert surgeries on Montesinos knots with \(|\pi_1K(r)| < \infty\) are completely classified.

**Theorem** [Meier ’12]
All Seifert surgeries on pretzel knots are completely classified.

“**Theorem”** [Ichihara-Masai ’13]
All Seifert surgeries on alternating knots are completely classified.
Proof of Theorem 2

**Theorem 2** [Ichihara-J. ’13(’09)]

Let $K$ be an alternating knot and $K(r)$ be a toroidal Seifert manifold. Then

- $K = T(2, \pm 3)$ and $r = 0$,
- or
- $K = T(2, p) \# T(2, q)$ and $r = 2(p + q)$ with $|p|, |q| \geq 3$.

**Claim 1**

Let $K$ be a composite alternating knot and $K(r)$ be a toroidal Seifert manifold. Then

- $K = T(2, p) \# T(2, q)$ and $r = 2(p + q)$ with $|p|, |q| \geq 3$.

**Claim 2**

Let $K$ be a prime alternating knot and $K(r)$ be a toroidal Seifert manifold. Then

- $K = T(2, \pm 3)$ and $r = 0$. 
Proof of Claims

Claim 1

\(K\): composite alternating knot, \(K(r)\): toroidal Seifert manifold

\[\Rightarrow K = T(2, p)\# T(2, q) \text{ and } r = 2(p + q) \text{ with } |p|, |q| \geq 3.\]

This follows from the classification of non-simple Seifert surgeries on non-hyperbolic knots by Miyazaki-Motegi. □
Proof of Claims

**Claim 1**

\[
K: \text{composite alternating knot}, \quad K(r): \text{toroidal Seifert manifold} \\
\Rightarrow K = T(2, p) \# T(2, q) \quad \text{and} \quad r = 2(p + q) \quad \text{with} \quad |p|, |q| \geq 3.
\]

This follows from the classification of non-simple Seifert surgeries on non-hyperbolic knots by Miyazaki-Motegi.

**Lemma 1** [Boyer-Zhang, Patton]

**Toroidal** surgeries on **alternating** knots are completely classified.

**Lemma 2** [Ichihara-Motegi-Song]

\[
K: \text{small hyperbolic knot} \quad r: \text{\partial-slope of} \ K \\
\text{If} \ K(r) \text{is a Seifert manifold with} \ |\pi_1 K(r)| = \infty, \\
\text{then} \ K \text{is fibered and} \ r = 0.
\]
Claim 2

\[ K : \text{prime} \text{ alternating knot}, \quad K(r) : \text{toroidal Seifert manifold} \]
\[ \Rightarrow K = T(2, \pm 3) \text{ and } r = 0. \]
Proof of Claims

Claim 2

\[ K : \text{prime alternating knot}, \quad K(r) : \text{toroidal Seifert manifold} \]
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By Lemma 1, \( K \) is either 2-bridge or pretzel with length 3.
Proof of Claims

Claim 2

$K : \text{prime alternating knot, } K(r) : \text{toroidal Seifert manifold}$

$\Rightarrow K = T(2, \pm 3) \text{ and } r = 0.$

By Lemma 1, $K$ is either 2-bridge or pretzel with length 3.

If $K$ is 2-bridge, then $K = T(2, \pm 3) \text{ and } r = 0.$
## Proof of Claims

### Claim 2

| $K$ : prime alternating knot, $K(r)$ : toroidal Seifert manifold | $K = T(2, \pm 3)$ and $r = 0$. |

By Lemma 1, $K$ is either 2-bridge or pretzel with length 3.

If $K$ is 2-bridge, then $K = T(2, \pm 3)$ and $r = 0$.

If $K = P(a, b, c)$ which is not a two-bridge knot, then $K$ is small. Therefore $K$ is fibered and $r = 0$ by Lemma 2.
Proof of Claims

Claim 2

\( K : \) prime alternating knot, \( K(r) : \) toroidal Seifert manifold

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If \( a, b, c \) are odd, then \( g(P(a, b, c)) = 1 \), contradicts to fiberedness.
Proof of Claims

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If \( a, b, c \) are odd, then \( g(P(a, b, c)) = 1 \), contradicts to fiberedness.

If \( a \) is even and \( b, c \) are odd, then \( r \) is a \( \partial \)-slope of a 1-punctured Klein bottle. Then \( r = 2(b + c) \), contradicts to \( r = 0 \).\hfill \Box