Cosmetic surgery and the $SL(2,\mathbb{C})$ Casson invariant for two-bridge knots

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Joint work with
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Two slopes for a knot $K$ are called **equivalent** if $\exists$ homeo. of the exterior of $K$ taking one slope to the other.

Two surgeries on $K$ are called **purely cosmetic** if $\exists$ orientation preserving homeo. between the manifolds obtained by the surgeries.
Cosmetic surgery conjecture

Two surgeries on inequivalent slopes are never purely cosmetic.

This is the Problem 1.81(A) in Kirby’s list.

Remark: There exists some example of knots admitting “chirally” cosmetic surgeries along inequivalent slopes.
Theorem 1. (2-bridge knots with at most 9 crossings)

All the two-bridge knots of at most 9 crossings other than $[9_{27}]$ admits no cosmetic surgery pairs.

Remark: $[9_{27}] = \mathcal{S}(49, 19) = C[2, 2, -2, 2, 2, -2]$
Cosmetic surgery on 2-brigde knots

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Culler-Shalen norm
Computing Boundary slope

Boyer-Lins (1990)

A knot $K$ satisfying $\Delta_{K}^\prime\prime(1) \neq 0$ has no cosmetic surgery pairs.

Remark:
$\Delta_{K}(t)$ denotes the (symmetrized) Alexander polynomial for $K$. They use the Casson invariant (original, $SU(2)$-version).
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Boyer-Lins (1990)

A knot $K$ satisfying $\Delta''_K(1) \neq 0$ has no cosmetic surgery pairs.

Remark:
$\Delta_K(t)$ denotes the (symmetrized) Alexander polynomial for $K$. They use the Casson invariant (original, $SU(2)$-version).

Ni-Wu (2011)

Let $K$ be a nontrivial knot in $S^3$ and $r_1, r_2 \in \mathbb{Q}$ two slopes. If the surgeries along $r_1$ and $r_2$ are purely cosmetic, then $r_1, r_2$ satisfy that

(a) $r_1 = -r_2$,
(b) $q^2 \equiv -1 \mod p$ for $r_1 = p/q$,
(c) $\tau(K) = 0$ (the invariant defined by Ozsváth-Szabó).

Remark: They use Heegaard Floer homology.
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Table: 2-bridge knots of at most 9 crossings with $\tau = 0$

<table>
<thead>
<tr>
<th>Name</th>
<th>Schubert Form</th>
<th>Alexander Polynomial</th>
<th>$\Delta_K''(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4_1</td>
<td>$S(5, 2)$</td>
<td>$t^{-1} - 3 + t$</td>
<td>2</td>
</tr>
<tr>
<td>6_1</td>
<td>$S(9, 7)$</td>
<td>$2t^{-1} - 5 + 2t$</td>
<td>4</td>
</tr>
<tr>
<td>6_3</td>
<td>$S(13, 5)$</td>
<td>$t^{-2} - 3t^{-1} + 5 - 3t + t^2$</td>
<td>2</td>
</tr>
<tr>
<td>7_7</td>
<td>$S(21, 8)$</td>
<td>$t^{-2} - 5t^{-1} + 9 - 5t + t^2$</td>
<td>-2</td>
</tr>
<tr>
<td>8_1</td>
<td>$S(13, 11)$</td>
<td>$3t^{-1} - 7 + 3t$</td>
<td>6</td>
</tr>
<tr>
<td>8_3</td>
<td>$S(17, 4)$</td>
<td>$4t^{-1} - 9 + 4t$</td>
<td>8</td>
</tr>
<tr>
<td>8_8</td>
<td>$S(25, 9)$</td>
<td>$2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2$</td>
<td>4</td>
</tr>
<tr>
<td>8_9</td>
<td>$S(25, 7)$</td>
<td>$t^{-3} - 3t^{-2} + 5t^{-1} - 7 + 5t - 3t^2 + t^3$</td>
<td>4</td>
</tr>
<tr>
<td>8_{12}</td>
<td>$S(29, 12)$</td>
<td>$t^{-2} - 7t^{-1} + 13 - 7t + t^2$</td>
<td>-6</td>
</tr>
<tr>
<td>8_{13}</td>
<td>$S(29, 11)$</td>
<td>$2t^{-2} - 7t^{-1} + 11 - 7t + 2t^2$</td>
<td>2</td>
</tr>
<tr>
<td>9_{14}</td>
<td>$S(37, 14)$</td>
<td>$2t^{-2} - 9t^{-1} + 15 - 9t + 2t^2$</td>
<td>-2</td>
</tr>
<tr>
<td>9_{19}</td>
<td>$S(41, 16)$</td>
<td>$2t^{-2} - 10t^{-1} + 17 - 10t + 2t^2$</td>
<td>-4</td>
</tr>
<tr>
<td>9_{27}</td>
<td>$S(49, 19)$</td>
<td>$t^{-3} - 5t^{-2} + 11t^{-1} - 15 + 11t - 5t^2 + t^3$</td>
<td>0</td>
</tr>
</tbody>
</table>

Remark:

For alternating knots, $\tau(K) = \sigma(K)$ (signature of $K$) holds.
Theorem 2. (A family including $9_{27}$)

Let $K_x$ be a 2-bridge knot $C[2x, 2 - 2x, 2x, 2, -2x]$ with $x \geq 1$. Then $K_x$ admits no cosmetic surgery pairs yielding homology 3-spheres.

i.e., any $\frac{1}{n}$- and $\frac{1}{m}$-surgeries are not purely cosmetic for $K_x$.

Remark:
For $K_x$, $\Delta''_{K_x}(1) = 0$ and $\tau(K_x) = 0$ hold. In particular, $K_1 = 9_{27}$.
Key Ingredient

**Definition (\(SL(2, \mathbb{C})\) Casson invariant) [very rough]**

For a closed orientable 3-manifold \(\Sigma\), the \(SL(2, \mathbb{C})\) Casson invariant \(\lambda_{SL(2, \mathbb{C})}(\Sigma)\) is defined by counting the (signed) equivalence classes of representations of the fundamental group in \(SL(2, \mathbb{C})\).

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Boden-Curtis (2012)

Let $K = S(\alpha, \beta)$ be a 2-brigde knot and $K(p/q)$ the 3-manifold obtained by $p/q$-surgery on $K$. Suppose that $p/q$ is not a strict boundary slope and no $p'$-th root of unity is a root of $\Delta_K(t)$, where $p' = p$ if $p$ is odd and $p' = p/2$ if $p$ is even. Then

$$\lambda_{SL(2, \mathbb{C})}(K(p/q)) = \begin{cases} \frac{1}{2} \|p/q\|_T & \text{if } p \text{ is even}, \\ \frac{1}{2} \|p/q\|_T - (\alpha - 1)/4 & \text{if } p \text{ is odd}. \end{cases}$$

Here $\|p/q\|_T$ denotes the total Culler-Shalen seminorm for $p/q$.

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Culler-Shalen norm & Ohtsuki’s method

Boden-Curtis, based on Ohtsuki (1994)

$$||p/q||_T = \frac{1}{2} \left( -|p| + \sum_i W_i \Delta(p/q, N_i) \right)$$

Here $N_1, \cdots, N_n$ denotes the boundary slope for $K$, and $W_i := \prod_j (|n_j| - 1)$ for the continued fraction expansion $[n_1, \cdots, n_m]$ associated to $N_i$.

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Mattman-Maybrun-Robinson (2008)

The boundary slopes of $S'(\alpha, \beta)$ are associated to the continued fractions obtained by applying the substitutions at non-adjacent positions in the simple continued fraction of $\alpha/\beta$.

Substitution 1:

$[b_0, 2b_1, b_2, b_3, \ldots, b_n] \mapsto [b_0 + 1, (-2, 2)^{b_1-1}, -2, b_2 + 1, b_3, \ldots, b_n]$

Substitution 2:

$[b_0, 2b_1 + 1, b_2, b_3, \ldots, b_n] \mapsto [b_0 + 1, (-2, 2)^{b_1}, -b_2 - 1, -b_3, \ldots, -b_n]$

The simple continued fraction is the unique one with all terms positive and greater than 1.