On a chirally cosmetic filling

In Dae Jong

Kindai University

joint work with
Kazuhiro Ichihara (Nihon University)

E-KOOK Seminar 2016
@Osaka Electro-Communication University
2016/8/24 10:00–10:30
Definition

\(X\) : an oriented compact 3-mfd. with \(\partial X = T^2\)

\(X(r)\) : the 3-mfd. obtained by \(r\)-Dehn filling on \(X\) \((r : \text{slope})\)

**Definition (cosmetic filling)**

\(r_1\)-filling and \(r_2\)-filling on \(X\) are **purely** (resp. **chirally**) cosmetic

\[\iff \exists h: X(r_1) \to X(r_2): \text{ori. preserving (resp. reversing) homeo.}\]
**Definition**

$X$ : an oriented compact 3-mfd. with $\partial X = T^2$

$X(r)$ : the 3-mfd. obtained by $r$-Dehn filling on $X$ ($r$ : slope)

**Definition (cosmetic filling)**

$r_1$-filling and $r_2$-filling on $X$ are purely (resp. chirally) cosmetic

$\overset{\text{def}}{\iff} \exists h : X(r_1) \to X(r_2)$: ori. preserving (resp. reversing) homeo.

- **mundane** $\overset{\text{def}}{\iff} \exists \phi : X \to X$: homeo. s.t. $\phi(r_1) = r_2$
- **exotic** $\overset{\text{def}}{\iff}$ not mundane
Definition

$X$: an oriented compact 3-mfd. with $\partial X = T^2$

$X(r)$: the 3-mfd. obtained by $r$-Dehn filling on $X$ ($r$: slope)

**Definition (cosmetic filling)**

$r_1$-filling and $r_2$-filling on $X$ are purely (resp. chirally) cosmetic  
\[\text{def} \iff \exists h: X(r_1) \to X(r_2): \text{ori. preserving (resp. reversing) homeo.} \]

- mundane $\iff \exists \phi: X \to X: \text{homeo. s.t. } \phi(r_1) = r_2$
- exotic $\iff$ not mundane

**ex.** Purely mundane cosmetic fillings: $\frac{1}{n} \cong \frac{1}{0}

**ex.** For all amphichiral knot $K$ and $\forall r \in \mathbb{Q}$,

$(\pm r)$-fillings on the exterior $E(K)$ are chirally mundane cosmetic.
Cosmetic Surgery Conjecture

Cosmetic Surgery Conj. [Problem 1.81(A)] in Kirby’s list

Cosmetic fillings are never purely and exotic.
Cosmetic Surgery Conjecture

Cosmetic Surgery Conj. [Problem 1.81(A)] in Kirby’s list

Cosmetic fillings are never purely and exotic.

Proposition [Mathieu ’92]

\[ X = E(\text{the trefoil knot}), \quad (k \geq 0) \]

\[ \Rightarrow \frac{18k + 9}{3k + 1} \quad \text{and} \quad \frac{18k + 9}{3k + 2} \]-fillings on \( X \) are chirally exotic cosmetic.

Remark

In Mathieu’s example, \( X \) and the resultants are Seifert.
Proposition [Bleiler-Hodgson-Weeks ’91]

\exists a hyperbolic knot \( K \subset S^2 \times S^1 \) s.t. \( E(K) \) admits chirally exotic cosmetic fillings yielding \( L(49, \pm 18) \).

\( (L(49, -18) = L(49, 19)) \)
**Proposition** [Bleiler-Hodgeon-Weeks ’91]

\[ \exists a \text{ hyperbolic knot } K \subset S^2 \times S^1 \text{ s.t. } E(K) \text{ admits chirally exotic cosmetic fillings yielding } L(49, \pm 18). \quad (L(49, -18) = L(49, 19)) \]

**Conjecture** [B.-H.-W. ’91] cf. Kirby’s list [Problem 1.81(B)]

Cusped hyperbolic manifolds admit NO exotic cosmetic fillings, **purely** or **chirally**, which yield hyperbolic manifolds.
Result (counterexample of [B.-H.-W.]’s conjecture)

Theorem [Ichihara-J. ’16]

\[ \exists \text{ a hyperbolic manifold admitting chirally exotic cosmetic fillings which yield hyperbolic manifolds.} \]
Construction - banding and the Montesinos trick -

$L \subset S^3$, $L'$: a link obtained from $L$ by a banding

Definition (chirally cosmetic banding)

A banding is chirally cosmetic $\iff L$ is ambient isotopic to $(L')!$
Construction - banding and the Montesinos trick -

$L \subset S^3$, $L'$: a link obtained from $L$ by a banding

\[ L \subset S^3, \quad L' : \text{a link obtained from } L \text{ by a banding} \]

\[ \xymatrix{ L \ar[rr]^{\text{banding}} & & L' } \]

**Definition (chirally cosmetic banding)**

A banding is **chirally cosmetic** $\iff L$ is ambient isotopic to $(L')!$

**Lemma (the Montesinos trick)**

\( K \): a knot in \( S^3 \) admitting a **chirally cosmetic banding**

\( \Sigma_K \): the double branched cover of \( S^3 \) branched along \( K \)

\( \implies \) We have **chirally cosmetic fillings** on the exterior of a knot in \( \Sigma_K \).
Construction - chirally cosmetic banding on $9_{27}$ -

**Proposition [Ichihara-J.]**

$9_{27} = S(49, 18)$ admits a **chirally cosmetic banding**.

**Remark**

- This is obtained from the *chirally exotic* cosmetic fillings of [B.-H.-W.] by using the Montesinos trick.
- $9_{27} = C(1, 1, 1, 2, -1, -1, -1, -2) = C(2, 2, -2, 2, 2, -2)$
Construction - chirally cosmetic banding on $9_{27}$ -

$9_{27}$ can be obtained as follows:
Construction - chirally cosmetic banding on $9_{27}$ -

This yields many chirally cosmetic banding by adding twists of the tangles, or increasing the number of tangles from $3 \times 2$ to $n \times 2$, ...
Construction - chirally cosmetic banding on $9_{27}$ -

This yields many chirally cosmetic banding by

- adding twists of the tangles, or
- increasing the number of tangles from $3 \times 2$ to $n \times 2$, ...
Construction - a generalization of $9_{27}$ -

Proposition [Ichihara-J.]
The knot $K$ admits a chirally cosmetic banding.
Construction - applying the Montesinos trick -
Construction

To show: $\Sigma_K$ and $\Xi := \overline{\Sigma}_K$ are hyperbolic, Cosmetic fillings (1/0- and 3-fillings) on $\overline{K}$ are exotic.
To show:

- $\Sigma_K$ and $X := \Sigma_K \setminus \bar{K}$ are hyperbolic,
- Cosmetic fillings (1/0- and 3-fillings) on $\bar{K}$ are exotic.
$\Sigma_K$ and $X$ are hyperbolic.

1. Using $\textit{SnapPy}$, we can obtain positively oriented triangulations of $\Sigma_K$, $X$, $X(1/0)$, and $X(3)$.

2. Then $\textit{hikmot}$ certifies these four manifolds are hyperbolic.

\[
\text{vol}(\Sigma_K) = 10.01776364\ldots, \\
\text{vol}(X) = 17.66121174\ldots, \\
\#(\text{tetrahedra of triangulation of } \Sigma_K) = 12, \\
\ldots\text{etc...}
\]

Reference

Lemma [Bleiler-Hodgson-Weeks ’91]

\( X : \) 1-cusped hyperbolic 3-mfd.

\( X \) is chiral \( \Rightarrow \) \( \exists \) self-homeo. on \( X \) changing a slope into the other.

\( \implies \) It suffices to show that \( X (= \Sigma_K \setminus \tilde{K}) \) is chiral.
1/0- & 3-fillings on $\bar{K}$ are exotic

**Lemma** [Bleiler-Hodgson-Weeks ’91]

$X$ : 1-cusped hyperbolic 3-mfd.
$X$ is chiral $\Rightarrow \nexists$ self-homeo. on $X$ changing a slope into the other.

$\Rightarrow$ It suffices to show that $X(= \Sigma_K \setminus \bar{K})$ is chiral.

1. Using a code introduced in [Dunfield-Hoffman-Licata], we can certify a given triangulation is canonical.

2. Then, using *SnapPy*, we can certify $X$ is chiral.

**Reference**

Result of our check (Thanks to Masai)

JongsMacBook:pyt InDaeJong$ python test.py
hikmot says
Manifold "positive_NewExDoubleBranchedCover.tri" is hyperbolic

hikmot says
Manifold "positive_NewExSD10.tri" is hyperbolic

hikmot says
Manifold "positive_NewExSD31.tri" is hyperbolic

hikmot says
Manifold "NewExSD.tri" is hyperbolic

SnapPy says
positive_NewExDoubleBranchedCover.tri is isometric to positive_NewExSD10.tri

SnapPy says
positive_NewExDoubleBranchedCover.tri is isometric to positive_NewExSD31.tri

Using code by Dunfield–Hoffman–Licata...
Is the triangulation of NewExSD.tri_filled canonical?
True
Is NewExSD.tri_filled amphicheiral?
False