Minimal coloring number for \( \mathbb{Z} \)-colorable links

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Knots in Tsushima, Sep 7, 2016
Let $L$ be a link, and $D$ a diagram of $L$.

A map $\gamma : \{\text{arcs of } D\} \to \mathbb{Z}$ is called a $\mathbb{Z}$-coloring on $D$ if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of $D$ with the over arc $a$ and the under arcs $b$ and $c$.

A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called the trivial $\mathbb{Z}$-coloring.
Example
Z-colorable link

$L$ is Z-colorable if $\exists$ a diagram of $L$ with a non-trivial Z-coloring.

Remark

$L$ is Z-colorable $\iff \det(L) = 0$

Any knot $K$ is non-Z-colorable since $\det(K)$ is odd.
Let \( L \) be a \( \mathbb{Z} \)-colorable link. Let us consider the cardinality of the image of a non-trivial \( \mathbb{Z} \)-coloring on a diagram of \( L \).

**Minimal coloring number**

We call the minimum of such cardinalities among all non-trivial \( \mathbb{Z} \)-colorings on diagrams of \( L \) the *minimal coloring number* of \( L \), and denote it by \( \mincol_{\mathbb{Z}}(L) \).
Let $L$ be a $\mathbb{Z}$-colorable link.

**Theorem 1**

If $L$ is non-splittable, then $\mincol_{\mathbb{Z}}(L) \geq 4$. 
Let $L$ be a $\mathbb{Z}$-colorable link.

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If $L$ is non-splittable, then $\text{mincol}_\mathbb{Z}(L) \geq 4$.

**Proposition**
If the crossing number of $L$ is at most 9, then $\text{mincol}_\mathbb{Z}(L) = 4$, i.e., $L$ can be colored by four colors.
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**Question**

How many colors are enough to color?
Example

Simple $\mathbb{Z}$-coloring
$\mathbb{Z}$-coloring by 5 colors
Simple $\mathbb{Z}$-coloring

Let $L$ be a $\mathbb{Z}$-colorable link, $\gamma$ a $\mathbb{Z}$-coloring on a diagram $D$ of $L$.

**Simple $\mathbb{Z}$-coloring**

We call $\gamma$ a simple $\mathbb{Z}$-coloring if $\exists d \in \mathbb{N}$ such that for all the crossings in $D$, the differences between the colors of the over arcs and the under arcs are $0$ or $d$. 

![Diagram of a link with numbers labeling crossings and arcs colored with numbers]

1. 2
2. 0
3. 1
4. 3
5. 2
6. 1
Theorem 2

Let \( L \) be a non-splittable \( \mathbb{Z} \)-colorable link. If there exists a simple \( \mathbb{Z} \)-coloring on a diagram of \( L \), then \( \mincol_{\mathbb{Z}}(L) = 4 \).
**Theorem 3**

If a non-splittable link $L$ admits a $\mathbb{Z}$-coloring $\gamma$ such that $\#\text{Im}(\gamma) = 5$, then $\text{mincol}_\mathbb{Z}(L) = 4$.

**Proposition**

If a $\mathbb{Z}$-coloring $\gamma$ satisfies $\#\text{Im}(\gamma) = 5$ and $\text{min\text{Im}}(\gamma) = 0$, then $\text{Im}(\gamma) = \{0, 1, 2, 3, 4\}, \{0, 1, 2, 3, 5\}, \{0, 1, 2, 3, 6\}, \{0, 1, 2, 4, 7\}, \{0, 2, 3, 4, 5\}, \{0, 3, 4, 5, 6\}$ or $\{0, 3, 5, 6, 7\}$, up to scale.
In the case $\text{Im}(\gamma) = \{0, 1, 2, 3, 5\}$
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In the case $\text{Im}(\gamma) = \{0, 1, 2, 4, 7\}$
Question

For any non-splittable $\mathbb{Z}$-colorable link $L$, $\text{mincol}_\mathbb{Z}(L) = 4$?

Question

Can any non-splittable $\mathbb{Z}$-colorable link admit a simple $\mathbb{Z}$-coloring?
Thank you for your attention.