Hyperbolic small knots in lens spaces

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Historical remarks (1)

In 1961/62, W. Haken introduced incompressible surface.
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$M$ : a compact orientable 3-manifold

$M$ is called sufficiently large if it contains a properly embedded (two-sided) incompressible surface. [Waldhausen, 1968]

$M$ is called Haken if it is irreducible and sufficiently large.
In 1968, F. Waldhausen showed the following by using hierarchy.

**Theorem (Topological rigidity).**

Let $M$ and $N$ be 3-manifolds. Suppose $M$ is Haken, and $\pi_1(M) \cong \pi_1(N)$ preserving peripheral structures. Then $M$ and $N$ are homeomorphic.

**Theorem.**

The universal cover of a Haken manifold is homeomorphic to $\mathbb{R}^3$. 
Historical remarks (3)

In 1970’s, W. Thurston showed the following by using hierarchy.

The Geometrization for Haken manifolds.

The interior of every Haken manifold has a canonical decomposition into pieces which have geometric structures.
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**The Geometrization for Haken manifolds.**

The interior of every Haken manifold has a canonical decomposition into pieces which have geometric structures.

In 2013, I. Agol showed the following.

**Virtual Haken Theorem**

Every compact, orientable, irreducible 3-manifold with infinite $\pi_1$ is virtually Haken, i.e., finitely covered by a Haken manifold.
Historical remarks (4)

Is every 3-manifold is Haken?
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No!

All but finitely many 3-manifold obtained by Dehn surgery on the figure-eight knot is non-Haken. [Thurston]

A. Hatcher, 1982

All but finitely many 3-manifold obtained by Dehn surgery on a small knot is non-Haken.
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Open Problem. [c.f. Rubinstein]

When 3-manifolds become Haken/non-Haken?
Definitions

Small manifold
An irreducible 3-manifold $M$ is called small if all the closed surfaces in $M$ are compressible or parallel to the boundary of $M$.

Small knot
A knot in a 3-manifold is called small if its exterior is small.
Examples

The following are examples of small knots.

- torus knot (in $S^3$ and lens spaces)
- 2-bridge knot in $S^3$ (Hatcher-Thurston)
- atoroidal genus one fibered knot
  (Floyd-Hatcher, Culler-Jaco-Rubinstein)
- Montesinos knot of length at most 3 (Oertel)
- some closed 3- or 4-braids (Boyer-Zhang)
- some knots in Sapphire spaces (Matsuda)
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Lopez Conjecture

Conjecture [Lopez, 1993] (c.f. Rubinstein)

Every closed orientable irreducible small 3-manifold contains a small knot.

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**Remark**

We can always find a non-hyperbolic knot in a small SFS. e.g., torus knot in lens space.
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**How about hyperbolic knots?**
**Lens space**

<table>
<thead>
<tr>
<th>“Theorem” [Lopez, 1993]</th>
</tr>
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<td>Every small Seifert fibered 3-manifold contains infinitely many hyperbolic small knots.</td>
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Lens space

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It was pointed out by H. Matsuda that its proof is incomplete (contains severe gap, remark after Conjecture 1.5, p.150).

**Lens space**

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**Theorem 1.**

Every lens space contains infinitely many hyperbolic small knot.

**Remark:** Also holds the same for $S^2 \times S^1$. 
Outline of Proof (1)

Let $L(p, q)$ be the lens space for coprime integers $p$ and $q > 0$. 
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Consider the following 2-bridge link $L_k = K \cup K'$.

The 2-bridge knot with $C(2, 2k, -2) \& S(4k - 1, 8k)$.

**Assumption:**

$4k \neq \pm p/q$
Performing Dehn surgery on $K' \subset L_k$ along the slope $-p/q$, we obtain $L(p, q)$ and a knot $K$ in it.

Lemma. The knot $K$ is a hyperbolic knot in $L(p, q)$.
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**Lemma.**

The knot $K$ is a hyperbolic knot in $L(p, q)$.

by the classification of exceptional surgeries on a component of 2-bridge link, obtained in [I., Archiv. Math. 2012].
It suffices to show that $K$ is a small knot.

Let $E(K)$ be the exterior of $K$ in $L(p, q)$. 
Outline of Proof (3)

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$\exists F$: closed incompressible surface of genus $\geq 2$ in $E(K)$. 

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$\exists F$: closed incompressible surface of genus $\geq 2$ in $E(K)$.

$\Rightarrow \exists F'$: essential surface with boundary on $\partial N(K')$ of slope $-p/q$ in $E(L_k)$. 
Perform meridional compressions on $F'$ for $K$. 
Outline of Proof (4)

⇒ Perform meridional compressions on $F'$ for $K$.

⇒ We have an essential surface in $E(L_k)$ with (possible) boundary on $K$ of slope $1/0$, and with boundary on $K'$ of slope $-p/q$. 
Outline of Proof (5)

Such surfaces, or precisely, such boundary slopes for 2-bridge links are completely classified, and \exists \text{algorithm} to compute them. [Floyd-Hatcher, Lash, Hoste-Shanahan]
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Boundary slope pairs for \( L_k \), [Hoste-Shanahan, §5, Table 4]

<table>
<thead>
<tr>
<th>( \partial )-slopes</th>
<th>restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>(0 \leq t \leq \infty)</td>
</tr>
<tr>
<td>((0, \phi), (\phi, 0))</td>
<td>(0 \leq t \leq \infty, k &gt; 1)</td>
</tr>
<tr>
<td>((-4k, \phi), (\phi, -4k))</td>
<td>(0 \leq t \leq 1)</td>
</tr>
<tr>
<td>((-4k, -2), (-2, -4k))</td>
<td>(1 \leq t \leq \infty)</td>
</tr>
<tr>
<td>((2t^{-1}, 2t))</td>
<td>(-1 \leq s \leq 1)</td>
</tr>
<tr>
<td>((-2t^{-1}, -2t))</td>
<td></td>
</tr>
<tr>
<td>((-2t^{-1} + 2 - 4k, -2t))</td>
<td></td>
</tr>
<tr>
<td>((-2t^{-1}, 2 - 4k - 2t))</td>
<td></td>
</tr>
<tr>
<td>((-1 - 2k + (2k - 1)s, -1 - 2k - (2k - 1)s))</td>
<td></td>
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Both \( t \) and \( s \) are rational parameters.
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**Spherical manifolds**

**Fact (c.f., [Seifert], …, [Perelman])**

3-manifolds with finite $\pi_1$ is either of type C, D, T, O, or I.

**Remark:** Such 3-manifolds are all small, and
- **C-type:** lens space
- **D-type:** prism manifold
**Spherical manifolds**

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**“Theorem” 2.**

“Most” spherical 3-manifolds of type T, O, or I contain hyperbolic small knots.
Strategy of Proof

Fact (c.f. [Moser], [Doig])

Up to orientation, any manifold of type $T$, $O$, or $I$ may be described as $(-1; (2, 1), (3, 1), (b_3, a_3))$, which is obtained by $(6a_3 - b_3)/a_3$-surgery on the right-hand trefoil.
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Note:

The trefoil is obtained by $(-1)$-surgery on $K' \subset L_1$. 
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Note:

The trefoil is obtained by $(-1)$-surgery on $K' \subset L_1$.

$\Rightarrow$ Similar arguments as before can (may) be applied.
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Remaining cases for finite $\pi_1$

Prism manifold $P(n, m)$

Seifert manifold: $(-1; (2, 1), (2, 1), (n, m))$ with base $S^2$.
Here $n > 0$ & $m$: coprime to $n$.

$(\text{twisted } I\text{-bundle} / \text{Klein bottle}) \cup_{(n,m)} (\text{Solid torus})$

Question.

Does every prism manifold contain small knots?
Further questions

Question.
Does every small Seifert manifold contain small hyperbolic knots?

Question.
Does every random manifold contain small hyperbolic knots?