Amphicheiral 3-manifolds not coming from amphicheiral null-homologous knot complements

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joint work with
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Amphicheiral manifold

\( M \): an oriented compact 3-manifold

**Definition**

\[ M : \text{amphicheiral} \iff \exists h : M \to M : \text{orientation reversing homeo}. \]
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**Examples**

- The 3-sphere \( S^3 \) is amphicheiral.

Remark: These examples are not hyperbolic.
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- The 3-sphere $S^3$ is amphicheiral.
- The lens space $L(p, q)$ is amphicheiral iff $q^2 \equiv -1 \mod p$.
  - $L(5, 1)$ is not amphicheiral (chiral). ($\therefore 1^2 = 1 \not\equiv -1 \mod 5$)
  - $L(5, 2)$ is amphicheiral. ($\because 2^2 = 4 \equiv -1 \mod 5$)
- For $\forall n \in \mathbb{N}$, $L(n^2 + 1, n)$ is amphicheiral.

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Amphicheiral hyperbolic manifold

\( N \): the magic manifold

**Remark**

Most of the hyperbolic mfds with small volume are obtained by Dehn filling on \( N \).

**Proposition** [Martelli-Petroino (2006)]

\[ \{ \text{amphi. 1-cusped hyp. mfd. obtained by Dehn filling on } N \} = \{ S^3 \setminus 4_1, \text{ its sibling} \} \]

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- \( K \) : amphicheiral hyperbolic knot in \( S^3 \)
  \( \Rightarrow S^3 \setminus K \) is amphicheiral 1-cusped hyperbolic 3-mfd.
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  \[ \Rightarrow S^3 \setminus K \text{ is amphichiral 1-cusped hyperbolic 3-mfd.} \]

Natural to ask the following

\( \exists \) amphichiral (1-cusped hyperbolic) manifold
not coming from amphichiral knot complements
Observation

$h: S^3 \rightarrow S^3$ : reflection, $K \subset S^3$ : amphi. knot, $p/q$ : slope for $K$
$\Rightarrow h(p/q) = -p/q$, and $\Delta(p/q, -p/q) = 2|pq| : \text{even.}$
Observation

\[ h : \mathbb{S}^3 \to \mathbb{S}^3 : \text{reflection, } K \subset \mathbb{S}^3 : \text{amphi. knot, } p/q : \text{slope for } K \Rightarrow h(p/q) = -p/q, \text{ and } \Delta(p/q, -p/q) = 2|pq| : \text{even}. \]

Definition

\[ M : \text{amphicheiral manifold, } h : M \to M : \text{ori.-rev. homeo.} \]

\[ K \subset M : \text{amphicheiral knot} \iff K \sim h(K) \text{ in } M. \]

Lemma

\[ M : \text{amphi. manifold, } h : M \to M : \text{ori.-rev. homeo.} \]

\[ K \subset M : \text{amphi. null-homologous knot, } p/q : \text{a slope for } K \Rightarrow h(p/q) = -p/q, \text{ and } \Delta(p/q, -p/q) = 2|pq| : \text{even}. \]
Amphicheiral knot complement

**Observation**

\[ h: S^3 \to S^3 : \text{reflection}, \ K \subset S^3 : \text{amphi. knot}, \ \frac{p}{q} : \text{slope for} \ K \Rightarrow h(\frac{p}{q}) = -\frac{p}{q}, \ \text{and} \ \Delta(\frac{p}{q}, -\frac{p}{q}) = 2|pq| : \text{even}. \]

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Q. \( \exists \) hyp. amphi. mfd. not coming from such knot complement?
Known example (figure-eight sibling)

$W$ : the figure-eight sibling (amphicheiral, 1-cusped hyperbolic)
**Known example (figure-eight sibling)**

\[ W : \text{the figure-eight sibling (amphicheiral, 1-cusped hyperbolic)} \]

\[ \exists \phi : W \rightarrow W : \text{orientation reversing homeo. s.t. } \phi(\infty) = -1 \]

\[ \Rightarrow W \not\cong \text{amphicheiral null-homologous knot complement} \]

since \[ \Delta(\infty, -1) = 1. \]
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$W(\infty) = L(-5, 1)$ & $W(-1) = L(5, 1)$. 
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Question

\( \exists \) other examples?
Main result

\[ T_n := \] 

\[ M_n := \text{(interior of) the double branched cover over } T_n \ (n \in \mathbb{Z}) \]

**Theorem [Ichihara-J.-Taniyama]**

- \( M_n \) is amphicheiral, and 1-cusped hyperbolic.
- \( M_n \not\cong \forall \) amphicheiral null-homologous knot complement in any closed amphicheiral 3-mfd.

**Remark:** \( M_2 = W \) (the figure-eight sibling)
The mirror image $T_n!$ is obtained from $T_n$ by $\frac{\pi}{2}$-rotation.
Amphicheirality

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The lift $\tilde{h}: M_n \to M_n$ is ori.-rev. homeo. $\Rightarrow M_n$ is amphicheiral.

Since $\tilde{h}(\tilde{\mu}) = \tilde{\lambda}$ and $\Delta(\tilde{\mu}, \tilde{\lambda}) = 1$, 

$M_n \not\cong \forall$ amphi. null-homologous knot complement.
Hyperbolicity – $K_n$ –

$K_n :=$

Lemma

$K_n$ is the two-bridge knot (or link) with Schubert’s normal form $S(n^4 - 2n^3 + 2n^2 - 2n + 1, n^3 - 2n^2 + n - 1)$.

Remark

$K_2 = \text{the torus knot of type } T(2, 5)$. 
Hyperbolicity – chirally cosmetic banding –

**Lemma**

\[ K_n \xrightarrow{\text{a banding}} K_n! \]

**Remark**

\[ T(2, 5) \xrightarrow{\text{a banding}} T(2, 5)! \] is found by [Zeković (2015)].
**Hyperbolicity – surgery description –**

Replace twists to bands

Take the double branched cover

Cut at \( \infty \)

Take the double branched cover

\[ -n \]

\[ n \]

\[ -n \]
Hyperbolicity – surgery description –

\[ n - n - n = 0 \]

\[ \Sigma_n := L(n^4 - 2n^3 + 2n^2 - 2n + 1, n^3 - 2n^2 + n - 1) \]

Lemma

\( M_n \) is a complement of a knot in \( \Sigma_n \).
Hyperbolicity – surgery description –

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**Lemma**

\( M_n \) is a complement of a knot in \( \Sigma_n \).

**Remark**

\( \Sigma_n \) is chiral (i.e. not amphicheiral) since

\[ (n^3 - 2n^2 + n - 1)^2 \equiv 1 \not\equiv -1 \mod (n^4 - 2n^3 + 2n^2 - 2n + 1). \]
Hyperbolicity – orientation reversing homeo –

slide

isotopy

slide

isotopy

twist on the gray component
Hyperbolicity

**Theorem** [Ichihara-J.-Taniyama]

\[ M_n(\infty) = \Sigma_n \text{ and } M_n(0) = -\Sigma_n. \]

**Lemma** [Matignon (2010)]

\[ \mathcal{L}: \text{a non-hyperbolic knot complement in a lens space} \]

If \( \mathcal{L}(\infty) \cong -\mathcal{L}(r) \), then \( r \neq 0 \).

\[ \Rightarrow M_n \text{ is hyperbolic. } \square \]
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**Summary**: \( M_n \) is
- (1-cusped) hyperbolic,
- amphicheiral,
- \( \not\cong \) to any amphicheiral null-homologous knot complement.
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**Problem**

\[ M_n \not\cong \forall \text{ amphicheiral null-homologous knot complement?} \]