$SL(2,\mathbb{C})$ Casson invariant and chirally cosmetic surgery

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Based on a joint work with

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Dehn surgery on a knot

\( K \): a knot (i.e., embedded circle) in a 3-manifold \( M \)

**Dehn surgery** on \( K \) (operation to produce a “NEW” 3-mfd)

1) remove the open neighborhood of \( K \) from \( M \)
   (to obtain the exterior \( E(K) \) of \( K \))

2) glue a solid torus back (along a slope \( \gamma \))

We denote the obtained manifold by \( M_K(\gamma) \),
or, by \( K(\gamma) \) if \( K \) is a knot in \( S^3 \).
Cosmetic surgery conjecture

It is natural to ask:

Can a pair of distinct Dehn surgeries give the same manifold?
Cosmetic surgery conjecture

It is natural to ask:

Can a pair of distinct Dehn surgeries give the same manifold?

Conjecture. (Problem 1.81(A) in Kirby’s list)

Two surgeries on inequivalent slopes are never purely cosmetic.

- Two slopes for a knot $K$ are called equivalent if $\exists$ homeo. of the exterior of $K$ taking one slope to the other.
- Two Dehn surgeries on $K$ are called purely cosmetic if $\exists$ orientation preserving homeo. between the manifolds obtained by the surgeries.
Chirally cosmetic case

For “Orientation reversing” case, there exist (counter-)examples.

[Mathieu, 1992]

There exist some knots admitting “chirally” cosmetic surgeries along inequivalent slopes.

In fact, \((18k + 9)/(3k + 1)\)- and \((18k + 9)/(3k + 2)\)-surgeries on the trefoil knot \(T_{2,3}\) in \(S^3\) yield orientation-reversingly homeomorphic pairs for any \(k \geq 0\).

Further examples were obtained by [Rong], [Matignon], [Bleiler-Hodgson-Weeks], [Hoffman-Matignon], [I.-Jong].
On amphicheiral knots

When $K$ is amphicheiral, for all slope $r \notin \{0, 1/0\}$, $r$- & $(-r)$-surgeries are chirally cosmetic along equivalent slopes.

Are other cosmetic surgeries on an amphicheiral knot in $S^3$:

If $K$ is amphicheiral and $K(r) \cong -K(r')$, then $K(-r') \cong -K(r') \cong K(r)$. By [Ni-Wu], this implies $r = \pm r'$. 
On amphicheiral knots

When $K$ is amphicheiral, for $\forall$ slope $r \notin \{0, 1/0\}$, $r$- & $(-r)$-surgeries are chirally cosmetic along equivalent slopes.

There are no other cosmetic surgeries on an amphicheiral knot in $S^3$:

If $K$ is amphicheiral and $K(r) \cong -K(r')$, then $K(-r') \cong -K(r') \cong K(r)$. By [Ni-Wu], this implies $r = \pm r'$.

Question

Can a non-trivial knot in $S^3$ admit (chirally) cosmetic surgeries other than torus knots and amphicheiral knots?
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**$SL(2, \mathbb{C})$ Casson invariant**

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**NOTE:** the inclusions $F \hookrightarrow W_i$, $W_i \hookrightarrow \Sigma$ induce surjections on $\pi_1$.  

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**NOTE:** the inclusions $F \hookrightarrow W_i$, $W_i \hookrightarrow \Sigma$ induce surjections on $\pi_1$.

### $X(N)$: the character variety for a manifold $N$

i.e., the set of characters of $SL(2, \mathbb{C})$ representations of $\pi_1(N)$.

Then we have the following diagram:

$$X(\Sigma) = X(W_1) \cap X(W_2) \to X(W_1)$$

$$\downarrow$$

$$X(W_2) \to X(F)$$

**NOTE:** $X(N)$ has the structure of complex affine algebraic set.
**$SL(2, \mathbb{C})$ Casson invariant**

$X^*(\Gamma)$: the subspace of characters of irreducible representations.
$SL(2, \mathbb{C})$ Casson invariant

$X^\ast(\Gamma)$: the subspace of characters of irreducible representations.

- For 0-dimensional components of $X^\ast(W_1) \cap X^\ast(W_2) \subset X^\ast(F)$, take a compact neighborhood $U$, disjoint from the higher dimensional components, and
- take an isotopy $h : X^\ast(F) \to X^\ast(F)$ supported in $U$ such that $h(X^\ast(W_1)) \cap X^\ast(W_2) \in U$. 

$\epsilon_{\chi} = \pm 1$, depending on whether the orientation of $h(X^\ast(W_1))$ followed by that of $X^\ast(W_2)$ agrees with the orientation of $X^\ast(F)$ at $\chi$. 

**Definition.** ($SL(2, \mathbb{C})$ Casson invariant) Define $\lambda_{SL(2, \mathbb{C})}(\Sigma) = \sum \epsilon_{\chi}$, where the sum is taken over all the 0-dimensional components of $X^\ast(W_1) \cap X^\ast(W_2)$. 


**SL(2, ℂ) Casson invariant**

$X^*(\Gamma)$: the subspace of characters of irreducible representations.

- For 0-dimensional components of $X^*(W_1) \cap X^*(W_2) \subset X^*(F)$, take a compact neighborhood $U$, disjoint from the higher dimensional components, and
- take an isotopy $h : X^*(F) \to X^*(F)$ supported in $U$ such that $h(X^*(W_1)) \cap X^*(W_2) \in U$.

Given a 0-dimensional component $\{\chi\}$ of $h(X^*(W_1)) \cap X^*(W_2)$, we set $\varepsilon_\chi = \pm 1$, depending on whether the orientation of $h(X^*(W_1))$ followed by that of $X^*(W_2)$ agrees with the orientation of $X^*(F)$ at $\chi$. 

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*Cosmetic surgery*
**SL(2, \mathbb{C}) Casson invariant**

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**Definition. (SL(2, \mathbb{C}) Casson invariant)**

Define $\lambda_{SL(2,\mathbb{C})}(\Sigma) = \sum \varepsilon_\chi$, where the sum is taken over all the 0-dimensional components of $h(X^*(W_1)) \cap X^*(W_2)$. 
Surgery formula

Surgery formula of $\lambda_{SL(2, \mathbb{C})}$

Suppose $K$ is a small knot in an integral homology 3-sphere $\Sigma$. Then, there exist $E_0, E_1 \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ depending only on $K$ such that for every admissible slope $p/q$, we have

$$\lambda_{SL(2, \mathbb{C})}(\Sigma_K(p/q)) = \frac{1}{2} \|p/q\|_{CS} - E_{\sigma(p)}.$$

Here $\|p/q\|_{CS}$ is the total Culler-Shalen semi-norm of the slope $p/q$ and $\sigma(p) \equiv p \pmod{2}$.
Total Culler-Shalen seminorm

Suppose $K$ is a small knot in an integral homology 3-sphere $\Sigma$ with complement $M$.

$\mathbf{f_\xi : X(M) \to \mathbb{C}}$ the regular function for $\xi \in H_1(\partial M) = \pi_1(\partial M)$

defined by $f_\xi = \chi(\xi) - 2$ for $\xi \in H_1(\partial M; \mathbb{Z})$.

$\mathbf{r : X(M) \to X(\partial M)}$ the map induced by $\pi_1(\partial M) \to \pi_1(M)$.

Let $\{X_i\}$ be the collection of all one-dimensional components of $X(M)$ such that $\dim r(X_i) = 1$ and $X_i \cap X^*(M) \neq \emptyset$. 
Total Culler-Shalen seminorm

\( f_{i, \xi} : X_i \to \mathbb{C} \) the regular function obtained by restricting \( f_\xi \) to \( X_i \).
Total Culler-Shalen seminorm

$$f_{i,\xi} : X_i \rightarrow \mathbb{C}$$ the regular function obtained by restricting $$f_\xi$$ to $$X_i$$.

For the smooth, projective curve $$\tilde{X}_i$$ birationally equivalent to $$X_i$$, denote the natural extension of $$f_{i,\xi}$$ to $$\tilde{X}_i$$ by $$\tilde{f}_{i,\xi} : \tilde{X}_i \rightarrow \mathbb{CP}^1$$. 
Total Culler-Shalen seminorm

\[ f_{i, \xi}: X_i \rightarrow \mathbb{C} \]
the regular function obtained by restricting \( f_\xi \) to \( X_i \).

For the smooth, projective curve \( \tilde{X}_i \) birationally equivalent to \( X_i \),
denote the natural extension of \( f_{i, \xi} \) to \( \tilde{X}_i \) by \( \tilde{f}_{i, \xi}: \tilde{X}_i \rightarrow \mathbb{C} \mathbb{P}^1 \).

For such \( X_i \), define the semi-norm \( \| \cdot \|_i \) on \( H_1(\partial M; \mathbb{R}) \) by setting
\[
\| \xi \|_i = \deg(\tilde{f}_{i, \xi})
\]for all \( \xi \) in the lattice \( H_1(\partial M; \mathbb{Z}) \).
**Total Culler-Shalen seminorm**

\[ f_{i,\xi} : X_i \rightarrow \mathbb{C} \]

the regular function obtained by restricting \( f_\xi \) to \( X_i \).

For the smooth, projective curve \( \tilde{X}_i \) birationally equivalent to \( X_i \), denote the natural extension of \( f_{i,\xi} \) to \( \tilde{X}_i \) by \[ \tilde{f}_{i,\xi} : \tilde{X}_i \rightarrow \mathbb{C} \mathbb{P}^1. \]

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\[ \| \xi \|_i = \text{deg}(\tilde{f}_{i,\xi}) \]

for all \( \xi \) in the lattice \( H_1(\partial M; \mathbb{Z}) \).

**Definition. (the total Culler-Shalen semi-norm)**

We define the total Culler–Shalen semi-norm of a slope \( p/q \) as

\[ \| p/q \|_{CS} := \sum_i m_i \| p/q \|_i \]

where \( m_i > 0 \) is the intersection multiplicity of \( X_i \) as a curve in the intersection \( X^*(W_1) \cdot X^*(W_2) \) in \( X(F) \).
Admissible slope

A slope $p/q$ on $\partial M$ is called admissible for a knot $K$ if

1. $p/q$ is a regular slope which is not a strict boundary slope;
2. No $p'$-th root of unity is a root of the Alexander polynomial of $K$, where $p' = p$ if $p$ is odd and $p' = p/2$ if $p$ is even.

Regular slope

A slope $\gamma$ on $\partial M$ is called regular if there are no irreducible representation $\rho : \pi_1(M) \to SL(2, \mathbb{C})$ satisfying that

1. the character $\chi_\rho$ lies on a one-dimensional component $X_i$ of $X(M)$ such that $r(X_i)$ is one-dimensional;
2. $\text{tr} \rho(\alpha) = \pm 2$ for all $\alpha$ in the image of $i^* : \pi_1(\partial M) \to \pi_1(M)$;
3. $\ker(\rho \circ i^*)$ is the cyclic group generated by $[\gamma] \in \pi_1(\partial M)$. 
CS semi-norm & $\partial$-slopes

$K$: a hyperbolic knot in an integral homology sphere $\Sigma$.

$\mathcal{B}_K = \{b_1/c_1, \ldots, b_m/c_m\}$: the set of boundary slopes for $K$

with $b_j/c_j \in \mathbb{Q}$, $b_j \geq 0$, and $b_1/c_1 < \cdots < b_m/c_m$. 

Proposition [Boyer-Zhang], [Mattman]

For a curve $X_i$ in the character variety $X(K)$, there exist non-negative constants $a_{ij} \geq 0$ depending only on $K$ such that

$$\|p/q\|_i = 2 \sum_{j=1}^m a_{ij} \Delta (p/q, b_j/c_j)$$

Here $\Delta(p/q, r/s) := |ps - rq|$: distance between slopes $p/q$, $r/s$, that is, the minimal geometric intersection number of the representatives of $p/q$ and $r/s$. 

CS semi-norm & ∂-slopes

\( K \) : a hyperbolic knot in an integral homology sphere \( \Sigma \).

\( B_K = \{ b_1/c_1, \ldots, b_m/c_m \} \): the set of boundary slopes for \( K \) with \( b_j/c_j \in \mathbb{Q}, b_j \geq 0, \) and \( b_1/c_1 < \cdots < b_m/c_m \).

Proposition [Boyer-Zhang], [Mattman]

For a curve \( X_i \) in the character variety \( X(K) \), there exist non-negative constants \( a^i_j \geq 0 \) depending only on \( K \) such that

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Here \( \Delta(p/q, r/s) := |ps - rq| \): distance between slopes \( p/q, r/s \), that is, the minimal geometric intersection number of the representatives of \( p/q \) and \( r/s \).
Total CS norm & \( \partial \)-slopes

Thus the total Culler-Shalen norm is given by

\[
\| p/q \|_{CS} = \sum_i m_i \| p/q \|_i = 2 \sum_i m_i \left( \sum_{j=1}^{m} a_j^i \Delta (p/q, b_j/c_j) \right)
\]

\[
= \sum_{j=1}^{m} \left( 2 \sum_i a_j^i m_i \right) \Delta (p/q, b_j/c_j).
\]

By putting \( w_j = 2 \sum_i a_j^i m_i \) we have the following formula:

\[
\| p/q \|_{CS} = \sum_{j=1}^{m} w_j |pc_j - qb_j|.
\]
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Result (1)

**Theorem 1.**

Let $K$ be a hyperbolic small knot in $\text{ZHS } \Sigma$. Assume that $\Sigma_K(p/q) \cong \pm \Sigma_K(p/q')$, and the slopes $p/q$ and $p/q'$ are admissible with $p/q, p/q' \not\in [b_1/c_1, b_m/c_m]$. Then;

(i) $qq' < 0$, i.e., the signs of slopes are opposite.

(ii) There exists a constant $C$ depending only on $K$ such that $(q + q')/p = C$.

(iii) If all the boundary slopes for $K$ are non-negative as rational numbers, the constant $C$ in (ii) equals to $\|\mu\|/2\|\lambda\|$, where $\mu$ denotes the meridional slope and $\lambda$ the preferred longitudinal slope for $K$. 
Result (2)

**Theorem 2.**

Let $K$ be a hyperbolic small knot in ZHS $\Sigma$. Assume that $\Sigma_K(p/q) \cong \pm \Sigma_K(p/q')$ and $p/q$ and $p/q'$ are admissible. If all the boundary slopes are non-negative (resp. non-positive), then $\frac{q+q'}{p} > 0$ (resp. $\frac{q+q'}{p} < 0$).

**Proof.**

We prove the case all the boundary slopes are non-negative.

Since $\Sigma_K(p/q) \cong \pm \Sigma_K(p/q')$, we have $||p/q|| = ||p/q'||$. By Theorem 1, it suffice to consider the case that $p/q' < 0 < p/q$. 
Result (2)

Let \( N \) be the integer that satisfies \( \frac{b_N}{c_N} < \frac{p}{q} < \frac{b_{N+1}}{c_{N+1}} \). When \( \frac{b_m}{c_m} < \frac{p}{q} \), we define \( N = m \). Then by Equation (1),

\[
\|p/q\| = \sum_{j=1}^{m} w_j \Delta(p/q, b_j/c_j) = \sum_{j=1}^{N} w_j (b_j q - pc_j) + \sum_{j=N+1}^{m} w_j (pc_j - b_j q).
\]

On the other hand, we have

\[
\|p/q'\| = \sum_{j=1}^{m} w_j (pc_j - b_j q').
\]
Result (2)

It follows from $\|p/q\| - \|p/q'\| = 0$ that

$$0 = \sum_{j=1}^{N} w_j b_j (q + q') - 2 \sum_{j=1}^{N} w_j c_j p + \sum_{j=N+1}^{m} w_j b_j (q' - q)$$

and so,

$$\frac{q + q'}{p} \sum_{j=1}^{N} w_j b_j = 2 \sum_{j=1}^{N} w_j c_j + \frac{q - q'}{p} \sum_{j=N+1}^{m} w_j b_j .$$

Since we are assuming all the boundary slopes are non-negative, i.e., $c_i \geq 0$, not all of $w_i$ are zero. Thus the right-hand side is always positive. This proves $\frac{q + q'}{p} > 0$. \qed
Application (1)

Theorem 3.
Let $K$ be an almost positive, hyperbolic small knot that satisfies;
all the boundary slopes of $K$ are non-negative.
If both $p/q$ and $p/q'$ are admissible, then $S^3_K(p/q) \not\cong -S^3_K(p/q')$.

Proof.
By Theorem 2, we have $\frac{q+q'}{p} > 0$.

On the other hand, by using (original) Casson invariant, together with properties of almost positive knots ([Przytycki-Taniyama], [Cromwell]), we have $\frac{q+q'}{p} < 0$. 
Application (2)

Theorem 4.

Let $K$ be a two-bridge knot of genus one. If the $r$- and $r'$-surgeries on $K$ are chirally cosmetic, then either

(i) $K$ is amphicheiral and $r = -r'$, or
(ii) $K$ is the positive or the negative trefoil, and

$$\{r, r'\} = \left\{\frac{18k + 9}{3k + 1}, \frac{18k + 9}{3k + 2}\right\}, \left\{\frac{-18k + 9}{3k + 1}, \frac{-18k + 9}{3k + 2}\right\} \quad (k \in \mathbb{Z}).$$
Double twist knots

Let $K$ be a two-bridge knot of genus one.

From [Hatcher-Thurston], $K$ must be represented as the double twist knot $J(\ell, m)$ with even $\ell, m$ & $\ell > 0$. 
Corollary

Let $K$ be a positive two-bridge knot with the Alexander polynomial $\Delta_K(t)$. If $\Delta_K(\zeta) \neq 0$ for any root of unity $\zeta$, then $K$ admits no chirally cosmetic surgeries.