Thin position for incompressible surfaces in 3-manifolds

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"alternating" thin position

Thin position argument:

Many good results, for knots & 3-manifolds.

$M$: a closed irreducible orientable 3-manifold
$F$: a separating closed orientable incompressible surface in $M$
$S$: a strongly irreducible Heegaard surface in $M$

Aim of this talk:
Put $F$ in a kind of thin position with respect to $S$. 
“alternating” thin position

Recall: a Heegaard splitting implies a singular foliation of $M$ by copies of $S$.

$S_t$: the level surface $(0 < t < 1)$

$H^t_1, H^t_2$: the handlebodies obtained by splitting $M$ along $S_t$ (As $t \to 0, 1$, $H^t_1$ or $H^t_2$ converges to a graph in $M$)

We can assume:

- No loop in $F \cap S_t$ bounds a disk on $S_t$.
- For $t$ small enough, $F \cap H^t_1$ consists of a family of meridian disks.
“alternating” thin position

\( M_+, M_- \): the two sides of \( F \)

**1st step**

Perform all possible boundary compressions of \( F \cap H^t_2 \) so that bands of \( S_t \) get pushed across \( F \) from \( M_- \) to \( M_+ \).

In a sense, the effect is to make \( S_t \cap M_- \) thin and \( S_t \cap M_+ \) thick.
“alternating” thin position

Fix this copy of $S_t$ as level one and denote it by $S_{t_1}$ with the initial $S$ as $S_{t_0}$ (where $F \cap H_{t_0}^1$ are meridian disks).

2nd step

Repeat the process for $H_{t_1}^2$ bounded by $S_{t_1}$, but this time interchanging the roles of $M_+, M_-$ so that bands of $S_{t_1}$ get pushed across $F$ from $M_+$ to $M_-$. This will give a new level surface $S_{t_2}$ for which $S_{t_2} \cap M_-$ is thick and $S_{t_2} \cap M_+$ thin.
We iterate until eventually $F$ meets a handlebody corresponding to $H^t_2$ in meridian disks only, for $t$ close to 1.

**Alternating thin position**

Call the obtained surface $F$ in an “alternating” thin position.

**Note:**

there are a finite number of critical levels $\hat{t}$, for $0 < \hat{t} < 1$, so that at such a level there is a single saddle critical point.
We can find at least one thin surface which is incompressible.

**Theorem.**

Let $F$, $S$, $M_+$, $M_-$ as above. Then either;

- there is a non-critical level $t$ so that $S_t \cap M_+$ is incompressible and $S_t \cap M_-$ has compressing disks on both sides of $S_t$, or the same with $M_+$, $M_-$ interchanged.

- there is a critical level $\hat{t}$ so that $S_t \cap M_+$ (resp. $S_t \cap M_-$) is incompressible for $t < \hat{t}$ (resp. $t > \hat{t}$) with $t$ close to $\hat{t}$, or the same with $M_+$, $M_-$ interchanged.

- there is a critical level $\hat{t}$ so that both $S_t \cap M_+$, $S_t \cap M_-$ are incompressible for $t > \hat{t}$ and $t$ arbitrarily close to $\hat{t}$. 

If we consider the Hempel distance of a Heegaard surface, we obtain the following corollary.

**Corollary 1.**

Under the same settings as in Theorem, suppose that $S$ has Hempel distance at least 4. Then only the third possibility in Theorem can occur.
Corollary

Corollary 2.

Let $M_+, M_-$ be compact orientable irreducible 3-manifolds with incompressible boundary $\partial M_+ \cong \partial M_-$. Amongst all incompressible and $\partial$-incompressible surfaces in $M_+, M_-$, choose the ones $A_+, A_-$ with $|\partial A_+| \geq |\partial A_-|$ which minimize

$$h = |\chi(A_+)| + |\chi(A_-)| + 2(2|\partial A_+| - |\partial A_-| - 1).$$

Then $h$ gives a lower bound for the absolute value of the Euler characteristic of a Heegaard surface of Hempel distance at least 4 in a closed 3-manifold obtained by gluing $M_+$ and $M_-$ along their boundary.