Achiral 1-cusped hyperbolic 3-mfds not coming from amphicheiral null-homologous knot complements

In Dae Jong

Kindai University

joint work with
Kazuhiro Ichihara (Nihon University)
Kouki Taniyama (Waseda University)

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Achiral manifold

$M$ : an oriented 3-manifold

**Definition**

$M$ is achiral $\iff \exists r : M \to M$ : orientation reversing self-homeo.

- The 3-sphere $S^3$ is achiral.
- A lens space $L(p, q)$ is achiral $\iff q^2 \equiv -1 \mod p$. 
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- A lens space $L(p, q)$ is achiral $\iff q^2 \equiv -1 \mod p$.
- **Achiral hyperbolic 3-manifolds are quite sporadic** (at least among those with small volume) [cf. Martelli-Petroino (’06)].
- Complements of amphicheiral hyperbolic knots in $S^3$ are achiral 1-cusped hyperbolic 3-manifolds.
Achiral manifold

\[ M : \text{an oriented 3-manifold} \]

**Definition**

\[ M : \text{achiral} \iff \exists r : M \to M : \text{orientation reversing self-homeo.} \]

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- A lens space \( L(p, q) \) is achiral \( \iff q^2 \equiv -1 \mod p \).
- **Achiral hyperbolic 3-manifolds are quite sporadic** (at least among those with small volume) [cf. Martelli-Petroino ('06)].
- Complements of amphicheiral hyperbolic knots in \( S^3 \) are achiral 1-cusped hyperbolic 3-manifolds.

**Question**

Find **achiral** (1-cusped) **hyperbolic** 3-manifolds NOT coming from amphicheiral knot complements.
Observation

$r: S^3 \rightarrow S^3$: reflection, $K \subset S^3$: amphi. knot, $p/q$: slope for $K$

$\Rightarrow r(p/q) = -p/q$, and $\Delta(p/q, -p/q) = 2|pq|: \text{even.}$
Complement of an amphicheiral knot

Observation
\[ r: S^3 \to S^3 : \text{reflection}, \ K \subset S^3 : \text{amphi. knot}, \ p/q : \text{slope for} \ K \]
\[ \Rightarrow r(p/q) = -p/q, \text{ and } \Delta(p/q, -p/q) = 2|pq| : \text{even}. \]

Definition
\[ M : \text{achiral 3-manifold,} \quad r: M \to M : \text{ori.-rev. self-homeo.} \]
\[ K \subset M : \text{amphicheiral knot} \iff K \sim r(K) \text{ in } M. \]

Lemma
\[ M : \text{achiral 3-manifold,} \quad r: M \to M : \text{ori.-rev. self-homeo.} \]
\[ K \subset M : \text{amphi. null-homologous knot,} \quad p/q : \text{a slope for } K \]
\[ \Rightarrow r(p/q) = -p/q, \text{ and } \Delta(p/q, -p/q) = 2|pq| : \text{even}. \]
Main result

\[ T_n := \text{(interior of) the double branched cover over } T_n \quad (n \in \mathbb{Z}) \]

**Theorem [Ichihara-J.-Taniyama]**

- \( M_n \) is achiral, and (1-cusped) hyperbolic.
- \( M_n \not\cong \forall \text{amphicheiral null-homologous knot complement in any closed achiral 3-manifold.} \)

**Remark:** \( M_2 \) = the figure-eight sibling
The mirror image \( m(T_n) \) is obtained from \( T_n \) by \( \frac{\pi}{2} \)-rotation \( \rho \).

\[ \tilde{h} = \tilde{\rho} \circ \tilde{m} : M_n \to M_n \text{ is ori.-rev. self-homeo.} \implies M_n \text{ is achiral.} \]
The mirror image $m(T_n)$ is obtained from $T_n$ by $\frac{\pi}{2}$-rotation $\rho$.

$\tilde{h} = \tilde{\rho} \circ \tilde{m} : M_n \to M_n$ is ori.-rev. self-homeo. \(\Rightarrow M_n\) is achiral.

Since $\tilde{h}(\tilde{\mu}) = \tilde{\lambda}$ and $\Delta(\tilde{\mu}, \tilde{\lambda}) = 1$,

$M_n \not\cong \forall$ amphicheiral null-homologous knot complement.
Proof of Theorem – Hyperbolicity –

A surgery description of $M_n$:

\[ \Sigma_n := L(n^4 - 2n^3 + 2n^2 - 2n + 1, n^3 - 2n^2 + n - 1) \]

- $M_n$ is a complement of a knot in $\Sigma_n$.
- $\Sigma_n$ is chiral (i.e. not achiral) since
  \[ (n^3 - 2n^2 + n - 1)^2 \equiv 1 \not\equiv -1 \mod (n^4 - 2n^3 + 2n^2 - 2n + 1). \]
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**Theorem** [Ichihara-J.-Taniyama]

\[ M_n(\infty) = \Sigma_n \text{ and } M_n(0) = -\Sigma_n. \]

**Lemma** [Matignon (2010)]

\[ L : \text{a non-hyperbolic knot complement in a lens space} \]

If $L(\infty) \cong -L(r)$, then $r \neq 0$.

\[ \Rightarrow M_n \text{ is hyperbolic}. \]