Cosmetic surgeries on knots

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Dehn surgery on a knot

Let $K$ be a knot (i.e., embedded circle) in a 3-manifold $M$

**Dehn surgery on $K$** (operation to produce a “NEW” 3-mfd)

1) Remove the open neighborhood of $K$ from $M$.
2) Glue a solid torus back (along a slope $\gamma$).

slope : isotopy class of non-trivial simple closed curve
Cosmetic surgery conjecture

It is natural to ask:

Can distinct Dehn surgeries give the same manifold?

Conjecture. [Problem 1.81(A) in Kirby’s list]

Two surgeries on inequivalent slopes are never purely cosmetic.

- Two slopes for a knot $K$ are called equivalent if there exists a homeomorphism of the exterior of $K$ taking one slope to the other.
- Two surgeries on $K$ are called purely cosmetic if there exists an orientation-preserving homeomorphism between the manifolds obtained by the surgeries.
Chirally cosmetic case

For “Orientation reversing” case, there exist (counter-) examples.

Fact. [Mathieu, 1992]

There exist knots admitting “chirally” cosmetic surgeries along inequivalent slopes.

Actually $\frac{18k+9}{3k+1}$- and $\frac{18k+9}{3k+2}$-surgeries on the trefoil knot $T_{2,3}$ in $S^3$ yield orientation-reversingly homeomorphic pairs for any $k \geq 0$.

Further examples were obtained by [Rong], [Bleiler-Hodgson-Weeks], [Matignon], [Hoffman-Matignon].
2-bridge knots

Proposition. (2-bridge knots with at most 9 crossings)

All the two-bridge knots of at most 9 crossings other than $9_{27} = C[2, 2, -2, 2, 2, -2]$ admits no cosmetic surgery pairs.

Theorem. [I.-Saito, 2018]

Any 2-bridge knot $C[2x, 2 - 2x, 2x, 2, -2x]$ with $x \geq 1$ admits no cosmetic surgeries yielding homology 3-spheres. i.e., any $\frac{1}{n}$- and $\frac{1}{m}$-surgeries are not purely cosmetic for $K_x$.

Remark: For $K_x$, the known restrictions cannot be applied; (original) Casson invariant & Heegaard Floer homology. Our advantage is to use $SL(2, \mathbb{C})$ Casson invariant.
Jones polynomial

Let $V_K(t)$ be the Jones polynomial of a knot $K$ in $S^3$.


If a knot $K$ satisfies either $V''_K(1) \neq 0$ or $V'''_K(1) \neq 0$, then $K$ admits no purely cosmetic surgeries.

**Corollary.**

The cosmetic surgery conjecture is true for all knots with no more than 11 crossings, except possibly

$$10_{33}, 10_{118}, 10_{146}, 11a_{91}, 11a_{138}, 11a_{285}, 11n_{86}, 11n_{157}.$$ 

This result was extended by T. Ito [to appear, CAG].
New Example

**Theorem.** [I.-Jong with H.Masai, To appear in Osaka J. Math.]

There exists a hyperbolic knot with chirally cosmetic surgeries along inequivalent slopes yielding **hyperbolic** manifolds.

In the proof, Hyperbolicity & Chirailty were checked by a computer program; **HIKMOT** [Exper.Math. 2016].

This gives a **counter-example** to:

**Conjecture.** [Bleiler-Hodgson-Weeks, 1999]

Any hyperbolic knot admits no purely/chirally cosmetic surgeries yielding hyperbolic manifolds.
2-bridge knots

Theorem. [I-Ito-Saito, preprint]

- Let $K$ be a 2-bridge knot of genus one. If the $r$- and $r'$-surgeries on $K$ are chirally cosmetic, then
  1. $K$ is amphicheiral and $r = -r'$, or
  2. $K$ is the positive or the negative trefoil, and for some $k \in \mathbb{Z}$,

  $$\{r, r'\} = \left\{ \frac{18k + 9}{3k + 1}, \frac{18k + 9}{3k + 2} \right\}, \left\{ -\frac{18k + 9}{3k + 1}, -\frac{18k + 9}{3k + 2} \right\}.$$ 

- Let $K$ be a positive 2-bridge knot with Alexander polynomial $\Delta_K(t)$ such that $\Delta_K(\zeta) \neq 0$ for any root of unity $\zeta$. Then $K$ admits no chirally cosmetic surgeries.

Question.

Can a non-torus, chiral knot admit chirally cosmetic surgeries?