A lower bound on the number of diagonals for polyhedra

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Let $M$ be a non-compact hyperbolic 3-manifold of finite volume.

**Question.**

Can we decompose $M$ into ideal tetrahedra?

**Theorem [Wada-Yamashita-Yoshida]**

Suppose that $M$ is obtained from $n$ convex ideal polyhedra $P_1, \ldots, P_n$ by identifying the faces in pairs. Suppose that each face of $P_i$ ($i = 1, \ldots, n - 1$) is pasted with a face of $P_n$, and the possibly remaining faces of $P_n$ glued in pairs. Then $M$ can be decomposed into non-degenerate ideal tetrahedra by subdividing the $P_i$'s.
Motivation

Let $P$ be a polyhedra, and denote

$$V = V(P), \quad E = E(P), \quad F = F(P), \quad F_d = F_d(P)$$

the sets of vertices, edges, faces, and $d$-gonal faces of $P$.

**Theorem 1 [Wada-Yamashita-Yoshida]**

In the notion above, we have

$$|V|(|V| - 1) \geq 8|F_4| + \sum_{d \geq 5} d(d - 1)|F_d|$$

Equality in the above holds if and only if $P$ is combinatorially equivalent to one of the polyhedra depicted in Figure 1.
Theorem 1 is equivalent to the following:

**Theorem 2 [Wada-Yamashita-Yoshida]**

Let $\Delta(P)$ denote the set of (interior) diagonals of a polyhedron $P$. Then the following always holds:

$$|\Delta(P)| \geq -\frac{3}{2} |F_3| + \sum_{d \geq 5} \frac{d}{2} |F_d|$$

Equality in the above holds if and only if $P$ is combinatorially equivalent to one of the polyhedra depicted in Figure 1.
(calculation)

(left side) = |\Delta(P)| = 2

(right side) = -\frac{3}{2}|F_3| + \sum_{d \geq 5} \frac{d}{2}|F_d|
= -\frac{3}{2} \times 2 + \frac{5}{2} \times 2 = 2
(calculation)

(left side) = |\Delta(P)| = 6

(right side) = \(-\frac{3}{2}|F_3| + \sum_{d \geq 5} \frac{d}{2}|F_d|\)

= \(-\frac{3}{2} \times 3 + \frac{5}{2} \times 3 + \frac{6}{2} \times 1\) = 6
Wada-Yamashita-Yoshida

In the proof of Theorem 1, (after smart reductions to the finite number of cases), they wrote:

“Next consider the case where $r < 9$. Running a computer program shows that there are twelve sequences ...”

However, no actual codes could be shown...
Remarks

Wada-Yamashita-Yoshida

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Our contribution

An alternative proof of Thm 2 without computer-asistance.
Outline of Proof

Theorem 2 (1)

\[ |\Delta(P)| \geq -\frac{3}{2}|F_3(P)| + \sum_{d \geq 5} \frac{d}{2}|F_d(P)| \]

We show it by induction about the number of faces \(|F(P)|\).
Refer the right-hand side as \(\delta(P)\);

\[ \delta(P) = -\frac{3}{2}|F_3(P)| + \sum_{d \geq 5} \frac{d}{2}|F_d(P)| \]

(i) \(|F(P)| = 4 \) (the minimal number of faces for polyhedra)
\(|\Delta(P)| = 0 \quad \delta(P) = -6 \)
Lemma

If $|\Delta(P)| = 0$, then $3|F_3(P)| > \sum_{d \geq 5} d|F_d(P)|$ holds.

We show Lemma by contraposition.

Suppose that $3|F_3(P)| \leq \sum_{d \geq 5} d|F_d(P)|$
Lemma

If $|\Delta(P)| = 0$, then $3|F_3(P)| > \sum_{d \geq 5} d|F_d(P)|$ holds.

We show Lemma by contraposition.

Suppose that $3|F_3(P)| \leq \sum_{d \geq 5} d|F_d(P)|$

Then, there exists an edge such as;

$\Rightarrow |\Delta(P)| \neq 0$
Now suppose $|\Delta| \geq \delta$ holds for any polyhedron with $|F| \leq k - 1$ for $k \geq 5$, and consider a polyhedron $P$ with $|F(P)| = k$. 
Now suppose $|\Delta| \geq \delta$ holds for any polyhedron with $|F| \leq k - 1$ for $k \geq 5$, and consider a polyhedron $P$ with $|F(P)| = k$.

**Key operation: collapsing a face**
The number of diagonals toward to each vertex of $d$-gon is $d(d - 3)$ at least.
(setting)
(the number of quadrangle adjacent to collapsed face ):=\(d_4\)
(the number of pentagon adjacent to collapsed face ):=\(d_5\)
(the number of d-gon adjacent to collapsed face ):=\(d_6\) \((d \geq 6)\)

(change by collapsing a face: \(P \leadsto P'\))
\[
|\Delta(P)| - d(d - 3) - d_5(d - 2) - 2d_6(d - 2) \geq |\Delta(P')|
\]
\[
|F_3(P)| + d_4 = |F_3(P')|
\]
\[
\sum_{d \geq 5} d|F_d(P)| - d - 5d_5 - d_6 = \sum_{d \geq 5} d|F_d(P')|
\]
(calculation)

\[ |\Delta(P)| \geq |\Delta(P')| + d(d - 3) + d_5(d - 2) + 2d_6(d - 2) \]

\[ = -\frac{3}{2}|F_3(P')| + \frac{1}{2} \sum d|F_d(P')| + d(d - 3) \]

\[ + d_5(d - 2) + 2d_6(d - 2) \]

\[ = -\frac{3}{2}|F_3(P)| - \frac{3}{2}d_4 + \frac{1}{2} \sum d|F_d(P)| - \frac{d}{2} - \frac{5}{2}d_5 - \frac{d_6}{2} \]

\[ + d_5(d - 2) + 2d_6(d - 2) \]

\[ \geq \delta(P) + d^2 - \frac{7}{2}d - \frac{3}{2}d_4 - \frac{5}{2}d_5 + 3d_5 - \frac{d_6}{2} + 6d_6 \]

\[ = \delta(P) + d(d - \frac{7}{2}) - \frac{3}{2}d_4 + \frac{1}{2}d_5 + \frac{11}{2}d_6 \]

\[ \geq \delta(P) + d(d - 5) + \frac{1}{2}d_5 + \frac{11}{2}d_6 \]
Some other cases

If there are triangles adjacent to a face, we can’t collapse the face. Then, we need some more operations.

Operation 1
Operation 2
Proof for the equality

$$|\Delta(P)| \geq -\frac{3}{2}|F_3(P)| + \sum_{d \geq 5} \frac{d}{2}|F_d(P)|$$

$$\Rightarrow$$ one of the polyhedra shown in Figure 1

(Proof)

We conform the change by collapsing a face;

$$|\Delta(P)| - \delta(P) \geq |\Delta(P')| - \delta(P') + \boxed{\text{.}}$$

Then

$$|\Delta(P')| - \delta(P') \geq 0$$

by the argument before, and so;

$$|\Delta(P)| - \delta(P) = 0 \Rightarrow \boxed{\text{.}} \leq 0.$$

We calculate in the case that we can collapse a face and a collapse face is triangle.

$$\boxed{\text{.}} = 1 - d_4 - d_5 + d_6 + \frac{2}{3}d_0$$
Case by case arguments..

(1) \(1 - d_4 - d_5 + d_6 + \frac{2}{3} d_0 = 0\)
   (i) \(d_4 = d_5 = d_6 = 1, d_0 = 0\)
   (ii) \(d_4 = 2, d_6 = 1, d_0 = 0\)
   (iii) \(d_5 = 2, d_6 = 1, d_0 = 0 \Rightarrow \text{(Figure 1 right)}\)

(2) \(1 - d_4 - d_5 + d_6 + \frac{2}{3} d_0 < 0\)
   (i) \(d_4 = 3, d_0 = 0\)
       \(d_0 = 2\)
   (ii) \(d_4 = 2, d_5 = 1, d_0 = 0\)
       \(d_0 = 2\)
   (iii) \(d_4 = 1, d_5 = 2, d_0 = 0 \Rightarrow \text{(Figure 1 left)}\)
       \(d_0 = 2\)
   (iv) \(d_5 = 3, d_0 = 0\)
       \(d_0 = 2\)
Thank you for your attention!