Most graphs are knotted

Kazuhiro Ichihara
Nihon University, College of Humanities and Sciences

Joint work with
Thomas Mattman (CSU, Chico)

“Musubime no Suuri”, Waseda University, Dec. 25, 2018
Graph and Embedding

Graph

an ordered pair $G = (V, E)$ comprising a set $V$ of vertices together with a set $E$ of edges.

We always assume that graphs are simple (no loops or multiple edges), and identify the combinatorial object with the associated 1-dim. cell complex.

Embedding of $G$ into $\mathbb{R}^3$
A graph is called *intrinsically knotted* (IK), if every tame embedding in $\mathbb{R}^3$ contains a non-trivially knotted cycle.
A graph is called *intrinsically knotted* (IK), if every tame embedding in $\mathbb{R}^3$ contains a non-trivially knotted cycle.

**We want to ask;**

Are “random” graphs intrinsically knotted?

What’s a “random” graph?
Random Graph

Let $|V(G)| = n$ denote the order or number of vertices of a graph, $N = \binom{n}{2}$ the number of edges in the complete graph of order $n$.

Models 1 & 2

1 (Erdős-Rényi) Choose a graph $G(n, M)$ uniformly at random from the set of labelled graphs with $n$ vertices and $M$ edges. There are $\binom{N}{M}$ such graphs and the probability of choosing a particular graph is $\binom{N}{M}^{-1}$.

2 (Gilbert) For each of the possible $N$ edges, we select it as an edge of the graph $G(n, p)$ independently with probability $p$. 
Random Graph (cont’d)

Models 2.5 & 3

2.5 If \( p = \frac{1}{2} \) in Gilbert’s model, then every one of the \( 2^N \) labelled graphs on \( n \) vertices is equally likely. The probability of choosing a particular labelled graph with \( |V(G)| = n \) is then \( 2^{-N} \).

3 (Unlabelled version of Model 2.5) Let \( \Gamma_n \) denote the number of unlabelled graphs on \( n \) vertices. Choose a graph from this set uniformly at random. The probability of choosing a particular unlabelled graph with \( |V(G)| = n \) is \( \Gamma_n^{-1} \).
We want to ask;

Are random graphs intrinsically knotted?
We want to ask;

Are random graphs intrinsically knotted?

Answer 1.
In Model 2.5 or 3, there is a constant $n_{IK}$ such that, when $n \geq n_{IK}$, MOST order $n$ graphs are intrinsically knotted (i.e., at least half of such graphs are IK).
We want to ask;

Are random graphs intrinsically knotted?

**Answer 1.**

In Model 2.5 or 3, there is a constant $n_{IK}$ such that, when $n \geq n_{IK}$, MOST order $n$ graphs are intrinsically knotted (i.e., at least half of such graphs are IK).

**Answer 2.**

In all four models, the probability that a graph is intrinsically knotted goes to one as the number of vertices increases.
Result 1

Theorem 1.

In Model 2.5 or 3, there is a constant $n_{IK}$ such that, when $n \geq n_{IK}$, MOST order $n$ graphs are intrinsically knotted (i.e., at least half of such graphs are IK).

We can show that $13 \leq n_{IK} \leq 18$, but leave open the question of the exact value of $n_{IK}$. 
Proposition.

A graph $G$ with $|V(G)| = n \geq 7$ and $|E(G)| \geq 5n - 14$ is IK.
A graph $G$ with $|V(G)| = n \geq 7$ and $|E(G)| \geq 5n - 14$ is IK.

[Mader, 1968]
If $|V(G)| = n \geq 7$ and $|E(G)| \geq 5n - 14$, then $G$ has a $K_7$ minor.

Since $K_7$ is IK [Conway-Gordon], any graph with a $K_7$ minor is IK. □
Proof of Thm 1. (Model 2.5)

We show that, if $n \geq 18$, then most graphs of order $n$ are IK.

Pair off each order $n$ graph $G$ with its complement $\overline{G}$.

At least one of these two has at least $\frac{1}{2} \binom{n}{2} = \frac{n(n-1)}{4}$ edges.

If $n \geq 18$, we see that $n(n-1)/4 > 5n - 14$.

By Proposition, $G$ or $\overline{G}$ is IK.
Result 2

Theorem 2.
In all four models, the probability that a graph is IK goes to 1 as the number of vertices increases.
Assume $0 < p \leq 1$ in Model 2.
The probability that a graph is not IK is bounded by the probability that it has at most $5n - 15$ edges:

$$\text{Prob}(G \text{ not IK}) \leq \text{Prob}(\|G\| \leq 5n - 15)$$

$$= \sum_{k=0}^{5n-15} \binom{N}{k} p^k (1-p)^{N-k} \leq e^{-2t^2N}.$$

The last inequality is due to Hoeffding, with $t = p - (5n - 15)/N$, and shows that the probability approaches 0 as $n$ goes to infinity.
Thank you for your attention!

I wish you
a Merry Christmas
and
a Happy New Year !!