Cosmetic banding on knots

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joint work with
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Geometric Topology of low dimensions
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**Bandaging**

$L$: a link in $S^3$

$b: I \times I \to S^3$, embedding s.t. $b(I \times I) \cap L = b(I \times \partial I)$

**Definition (banding)**

$L' = (L - b(I \times \partial I)) \cup b(\partial I \times I)$ is called the link obtained from $L$ by a banding along the band $b$. 

![Diagram of banding process](image)
Bandings

\[ L : \text{a link in } S^3 \]
\[ b: I \times I \to S^3, \text{ embedding s.t. } b(I \times I) \cap L = b(I \times \partial I) \]

**Definition (banding)**

\[ L' = (L - b(I \times \partial I)) \cup b(\partial I \times I) \] is called the link obtained from \( L \) by a banding along the band \( b \).

**Question**

For a link \( L \), does a banding give the same link?
Cosmetic banding

$L : \text{a link in } S^3$,  \hspace{1cm}  $L' : \text{a link obtained from } L \text{ by a banding}$

Definition (cosmetic banding)

A cosmetic banding \( \iff \exists h : S^3 \rightarrow S^3 \), homeo., \( h(L) = L' \).
Cosmetic banding

$L$ : a link in $S^3$, $L'$ : a link obtained from $L$ by a banding

Definition (cosmetic banding)

A cosmetic banding $\overset{\text{def}}{\iff} \exists h : S^3 \to S^3$, homeo., $h(L) = L'$.

Example:

![Cosmetic Banding Diagram](image)

![Non-Cosmetic Banding Diagram](image)
Pure/Chiral

$L \subset S^3$,  
$L'$: a link obtained from $L$ by a cosmetic banding  
i.e., $\exists h: S^3 \to S^3$, homeo. s.t. $h(L) = L'$.

\textbf{Definition (purely / chirally cosmetic)}

A banding is purely (resp. chirally) cosmetic  
$\iff h$ is orientation preserving (resp. reversing).
Pure/Chiral

\[ L \subset S^3, \quad L': \text{a link obtained from } L \text{ by a } \text{cosmetic} \text{ banding} \]
i.e., \( \exists h: S^3 \to S^3, \text{ homeo. s.t. } h(L) = L' \).

**Definition (purely / chirally cosmetic)**

A banding is purely (resp. chirally) cosmetic
\[ \iff h \text{ is orientation preserving (resp. reversing)}. \]

**The aim of this talk:**

- Find various (purely or chirally) cosmetic bandings.
- For a given cosmetic banding, reveal the reason why the banding is cosmetic.
banding and 4-move

\[ \text{equivalent to} \]

4-move
**Lemma**

A banding yields a 4-move.
Pretzel knots (symmetric unions)

- $(n, 2, -n)$-pretzel knot

$P(n, 2, -n)$

$P(n, -2, -n)$

$\pi$-rot.

mirror image

$P(-n, -2, n)$
**Proposition** [Zeković ’14]

The (2, 5)-torus knot admits a cosmetic banding.
(2,5)-torus knot

Proposition [Zeković ’14]
The (2, 5)-torus knot admits a cosmetic banding.

▶ This banding cannot be realized by a 4-move.
(shown by using signature)
Figure-eight sibling

Remark

- The complement of the red colored knot in $L(5,1)$ is called the “figure-eight sibling” which is amphicheiral.
Generalization

Let $K_n$ be the link in $S^3$ shown in the figure below.

**Proposition** [I.-Jong-Taniyama, ’18]

Each of $K_n$ admits a chirally cosmetic banding ($n \neq 0, 1$). In particular, $K_2 = T(2, 5)$ does.
Bleiler-Hodgson-Weeks’s example

**Proposition** [Bleiler-Hodgson-Weeks, ’99]

∃ a hyperbolic knot $K \subset S^2 \times S^1$ s.t. $E(K)$ admits chirally exotic cosmetic fillings yielding $L(49, \pm 18)$. ($L(49, -18) = L(49, 19)$)
On $9_{27}$

**Observation**

$9_{27} = S(49, 18)$ admits a chirally exotic cosmetic banding.

![Diagram of knot with chirally exotic cosmetic banding]
On $9_{27}$

$9_{27}$ can be obtained as follows:

![Diagram of the knot transformation](image-url)
On $9_{27}$

mirroring & $2\pi/3$-rot.

0-slope banding
On $9_{27}$

This yields many chirally cosmetic banding by

- adding twists of the tangles, or
- increasing the number of tangles from $3 \times 2$ to $n \times 2$, ...
Generalization

Theorem [I.-Jong-Masai, ’18]

The hyperbolic knot $K$ admits a chirally exotic cosmetic banding.
Related to Cosmetic surgery conjecture

By using the computer-programs SnapPy and hikmot, it is shown that

- $\Sigma_K$ and $\Sigma_K \setminus \bar{K}$ are hyperbolic.
- The slopes 3 & 1/0 on $\partial N(\bar{K})$ in $\Sigma_K$ are inequivalent.
New example

Allison H. Moore, Mariel Vazquez
A note on band surgery and the signature of a knot
preprint, arXiv:1806.02440

(A) The knots 8_8 and 8_20.
(B) Chirally cosmetic banding along 8^*_8 and 8_20.

**Figure 4.** (A) The knots 8_8 and 8_20 are symmetric unions, and as such their determinants are squares. (B) Chirally cosmetic bandings relating the pair 8^*_8 and 8_8 (left) and the pair 8_20 and 8^*_20 (right). The banding exhibited for 8^*_8 was discovered via computer simulation as described in section 4.3. The banding for 8_20 induces a “4-move.”