Complete exceptional surgeries on two-bridge links

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joint work with
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Friday Seminar on Knot Theory
@Osaka City University
2019/11/15

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(Essential branched surface & allowable path)

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Dehn surgery

\( K \): a knot in \( S^3 \)

\( E(K) \): the exterior of \( K \) (i.e., \( S^3 \setminus N^\circ(K) \))

\( K(r) \): the manifold obtained by Dehn surgery on \( K \) along \( \gamma = r \).

Dehn surgery & slopes for a LINK are defined in the same way.

\( L = K_1 \cup K_2 = \Rightarrow L(r_1, r_2), L(r_1, *), L(*, r_2) \)

\( \uparrow \) complete surgery
Dehn surgery

*K*: a knot in $S^3$  

$E(K)$: the exterior of $K$  
(i.e., $S^3 \setminus N^\circ(K)$)

Dehn surgery: Gluing a solid torus to $E(K)$

$\gamma = [f(m)]:$ surgery slope identified with $r \in \mathbb{Q} \cup \{1/0\}$.

$K(r):$ the manifold obtained by Dehn surgery on $K$ along $\gamma = r$. 
Dehn surgery

\( K \): a knot in \( S^3 \)

\( E(K) \): the exterior of \( K \) (i.e., \( S^3 \setminus N^\circ(K) \))

**Dehn surgery:** Gluing a solid torus to \( E(K) \)

\( \gamma = [f(m)] \): surgery slope identified with \( r \in \mathbb{Q} \cup \{1/0\} \).

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Dehn surgery & slopes for a LINK are defined in the same way.

\[ L = K_1 \cup K_2 \implies L(r_1, r_2), \quad L(r_1, *), \quad L(*, r_2) \]

↑ complete surgery
Exceptional surgery

Hyperbolic Dehn surgery theorem [Thurston]
On each component of a hyperbolic link, there are only finitely many Dehn surgeries yielding non-hyperbolic manifolds.

Exceptional surgery
Dehn surgery on a hyperbolic link yielding a non-hyperbolic manifold

Ultimate goal
Classify all exceptional surgeries on hyperbolic links in $S^3$. 
Exceptional surgery

Hyperbolic Dehn surgery theorem [Thurston]
On each component of a hyperbolic link, there are only finitely many Dehn surgeries yielding non-hyperbolic manifolds.

Exceptional surgery
Dehn surgery on a hyperbolic link yielding a non-hyperbolic manifold

Ultimate goal
Classify all exceptional surgeries on hyperbolic links in $S^3$.

Target: 2-bridge links
2-bridge link

\[ [a_1, \ldots, a_k] = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots \frac{1}{a_k}}} \quad (k \text{ is odd})} \]

\[ L[a_1, \ldots, a_k] \]

\[ [a_1, \ldots, a_k] = p/q \]

We also denote the link by \( L_{p/q} \) if \([a_1, \ldots, a_k] = p/q\).
Known results & Main Theorem

- 2-bridge knots [Brittenham-Wu]
- Montesinos knots, alternating knots [Ichihara-Masai]
- a component of two-bridge links [Ichihara]
Known results & Main Theorem

- 2-bridge knots [Brittenham-Wu]
- Montesinos knots, alternating knots [Ichihara-Masai]
- A component of two-bridge links [Ichihara]

$L$: a hyperbolic 2-bridge link

Complete exceptional surgery

Dehn surgery on $L$ along the slopes $(r_1, r_2)$ s.t. $r_i \neq 1/0$ and $L(r_1, \ast)$ and $L(\ast, r_2)$ are hyperbolic & $L(r_1, r_2)$ is non-hyperbolic

Main Theorem [Ichihara-J.-Masai]

If Dehn surgery on $L$ along $(r_1, r_2)$ is complete exceptional, then $L$ & $(r_1, r_2)$ are equivalent to one of those given in the next four pages.

cf. [Goda-Hayashi-Song]

Reducible Dehn surgery on $L$ along $(r_1, r_2)$ are classified.
Main Theorem

\[-\frac{1}{m} \quad \frac{1}{n}\]  \[m \geq 1, \ n \neq 0, 1\]

<table>
<thead>
<tr>
<th>Link (a-1)</th>
<th>slopes ((r_1, r_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_{[3,3]})</td>
<td>((-2, -2) \quad (-2, -1) \quad (-1, -4) \quad (-1, -3) \quad (-1, -1) \quad (5, \frac{4}{3}))</td>
</tr>
<tr>
<td>(L_{[3,2n-1]})</td>
<td>((n - 2, n - 2) \quad (n + 3, \frac{2n-1}{2}))</td>
</tr>
<tr>
<td>(L_{[2m+1,-3]})</td>
<td>((m - 3, \frac{2m+1}{2}) \quad (m + 2, m + 2))</td>
</tr>
<tr>
<td>(L_{[2m+1,3]})</td>
<td>((m - 1, m - 1) \quad (m + 4, \frac{2m+1}{2}))</td>
</tr>
<tr>
<td>(L_{[2m+1,-5]})</td>
<td>((m - 5, m))</td>
</tr>
<tr>
<td>(L_{[5,2n-1]})</td>
<td>((n, n + 5))</td>
</tr>
<tr>
<td>(L_{[2m+1,5]})</td>
<td>((m + 1, m + 6))</td>
</tr>
<tr>
<td>(L_{[2m+1,2n-1]})</td>
<td>((m + n - 2, m + n + 2) \quad (\frac{2m+2n-1}{2}, \frac{2m+2n+1}{2}))</td>
</tr>
</tbody>
</table>
Main Theorem

\[-\frac{1}{m} - \frac{1}{l} - \frac{1}{n} - \frac{1}{m} - \frac{1}{n} - \frac{1}{l}\]

\((b-1)\) \hspace{2cm} \((b-2)\)

\((m \geq 1, \ |n| \geq 2, \ |l| \geq 2)\) \hspace{2cm} \((m \geq 1, \ |n| \geq 2, \ l \geq 1)\)

\[
\begin{array}{|c|c|}
\hline
\text{Link (b-1)} & \text{slopes } (r_1, r_2) \\
\hline
L_{[2,2n,2l]} & (l - 1, l - 1) \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\text{Link (b-2)} & \text{slopes } (r_1, r_2) \\
\hline
L_{[2,2n-1,-2l]} & (-l - 1, -l - 1) \\
L_{[2m,2n-1,-2]} & (m + 1, m + 1) \\
\hline
\end{array}
\]
Main Theorem

\[
\frac{-1}{m} - \frac{1}{n} - \frac{1}{l} - \frac{1}{m} - \frac{1}{n} - \frac{1}{l}
\]

\[(b-3) \quad (m \geq 1, \ |n| \geq 2, \ l \geq 1)\]

\[(b-4) \quad (m \geq 1, \ n \neq 0, \ l \neq 0, 1)\]

<table>
<thead>
<tr>
<th>Link (b-3)</th>
<th>slopes ((r_1, r_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_{[2,2n+1,2]})</td>
<td>((-3, -1) (-2, -2) (-2, -1) (-1, -4) (-1, -1))</td>
</tr>
<tr>
<td>(L_{[2,2n+1,2l]})</td>
<td>((l - 1, l - 1))</td>
</tr>
<tr>
<td>(L_{[2m,2n+1,2]})</td>
<td>((m - 1, m - 1))</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Link (b-4)</th>
<th>slopes ((r_1, r_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_{[3,2,3]})</td>
<td>((-3, -1) (-2, -2) (-2, -1) (-1, -4) (-1, -1))</td>
</tr>
<tr>
<td>(L_{[3,2,2l-1]})</td>
<td>((l - 2, l - 2))</td>
</tr>
<tr>
<td>(L_{[2m+1,2,3]})</td>
<td>((m - 1, m - 1))</td>
</tr>
<tr>
<td>(L_{[2m+1,2,2l-1]})</td>
<td>((l + m, l + m) (l + m, l + m + 1))</td>
</tr>
<tr>
<td>(L_{[2m+1,-2,-3]})</td>
<td>((m + 2, m + 2))</td>
</tr>
<tr>
<td>(L_{[2m+1,-2,2l-1]})</td>
<td>((l + m - 1, l + m) (l + m, l + m))</td>
</tr>
<tr>
<td>(L_{[2m+1,2n,2l-1]})</td>
<td>((N - 2, N + 2) (N - 1, N + 1) (N, N) (N = l + m + n))</td>
</tr>
</tbody>
</table>
Main Theorem

\[
\text{Link (c-1)} \quad \text{slopes } (r_1, r_2)
\]

\[
\begin{array}{c}
L_{[3,2,2,2l-1]} \quad (l-2, l-2)
\end{array}
\]

\[(m \geq 1, \ n \neq 0, \ l \neq 0, 1)\]
A part of Main Theorem (Theorem A)

Theorem A. [Ichihara-J.-Masai]

$L$: a hyperbolic 2-bridge link in $S^3$
$L$ admits a complete exc. surg. $\Rightarrow L \simeq$ one of the following:

(a-1) $L_{[2m+1,2n-1]}$ with $m \geq 1$, $n \neq 0, 1$.

(b-1) $L_{[2m,2n,2l]}$ with $m \geq 1$, $|n| \geq 2$, $|l| \geq 2$.

(b-2) $L_{[2m,2n-1,-2l]}$ with $m \geq 1$, $|n| \geq 2$, $l \geq 1$.

(b-3) $L_{[2m,2n+1,2l]}$ with $m \geq 1$, $|n| \geq 2$, $l \geq 1$.

(b-4) $L_{[2m+1,2n,2l-1]}$ with $m \geq 1$, $n \neq 0$, $l \neq 0, 1$.

(c-1) $L_{[2m+1,2n,-2\text{sgn}(l),2l-1]}$ with $m \geq 1$, $n \neq 0$, $l \neq 0, 1$.

(c-2) $L_{[2m+1,2n-1,-2\text{sgn}(l),2l]}$ with $m \geq 1$, $n \neq 0, 1$, $l \neq 0$.

Outline of the proof of Main Theorem:

1. Theorem A is proved by using essential branched surfaces.
2. Based on Theorem A, by using a computer, we can find candidates of surgery slopes, and then we obtain Main Theorem.
A **branched surface** $\Sigma$ is a union of finitely many compact smooth surfaces s.t. each point has a disk-neighborhood or a neighborhood as follows:

\[ \partial_v N(\Sigma) : \text{the vertical boundary} \quad \partial_h N(\Sigma) : \text{the horizontal boundary} \]

The central curve of $\partial_v N(\Sigma)$ is called a **cusp** of $\Sigma$. 
Lemma 1. [Wu (1998)]

$L = K_1 \cup K_2 :$ a hyperbolic link in $S^3$, $V_i := N(K_i)$.
If $\exists \Sigma :$ an essential branched surface in $E(L)$ s.t. each $V_i$ contains THREE meridional cusps as a connected component of $S^3 \setminus \text{int}N(\Sigma)$, 
$\Rightarrow L$ admits no complete exceptional surgery.
Allowable path $\Rightarrow$ essential branched surface

$L_{p/q}$: a 2-bridge link

If we have an “allowable path” for $p/q$ in the Farey diagram, then we can construct an essential branched surface in $E(L_{p/q})$ [Delman].
The diagram $D(p/q)$

$L_{p/q}$ : a 2-bridge link, \( p/q = [b_1, \ldots, b_k] \) (\( b_i \): even)

1. For each \( b_i \), construct a fan \( F_{b_i} \).

2. Glue the fans so that \( e' \) of \( F_{b_i} \) is glued to \( e \) of \( F_{b_{i+1}} \).
   - If \( b_i b_{i+1} < 0 \) \( \Rightarrow \) \( F_{b_i} \cap F_{b_{i+1}} \) is an edge.
   - If \( b_i b_{i+1} > 0 \) \( \Rightarrow \) \( F_{b_i} \cap F_{b_{i+1}} \) is a 2-simplex.

3. Mark * for vertices of type odd/odd.

ex. The diagram $D(12/31)$ for \([2, -2, -4, -2] = 12/31\)
Channel & allowable path in $D(p/q)$

Channels:

allowable path

A path $\overset{\text{def}}{\leftrightarrow}$ a connected union of edges or channels in $D(p/q)$

A path for $p/q$ $\overset{\text{def}}{\leftrightarrow}$ a path from $1/0$ to $p/q$

A path $\gamma$ is allowable $\overset{\text{def}}{\leftrightarrow}$ the following 3 conditions hold.

1. $\gamma$ passes any point of $D(p/q)$ at most once.

2. Other than the middle points of channels, $\gamma$ intersects the interior of at most one edge of any given simplex.

3. $\gamma$ contains at least one channel.
Allowable path $\Rightarrow$ ess. branched surface

Example Allowable paths for $12/31$ in $D(12/31)$
Allowable path $\Rightarrow$ ess. branched surface

Example: Allowable paths for $12/31$ in $D(12/31)$

Lemma 2. [Delman]

$\exists$ an allowable path $\gamma$ for $p/q$ containing $k$ channels
$\Rightarrow$ $\exists$ essential branched surface $\Sigma_\gamma$ in $E(L_{p/q})$.
Furthermore for $N(L_{p/q}) = V_1 \cup V_2 \subset S^3 \setminus \text{Int} N(\Sigma_\gamma)$, each of $V_i$ is a solid torus with $k$ meridional cusps.

Lemma 3. (obtained by combining Lemmas 1 and 2)

If there exists an allowable path for $p/q$ containing 3 channels, then $L_{p/q}$ admits no complete exceptional surgery.
Find an allowable path

\[ p/q = [b_1, \ldots, b_k] \quad (b_i : \text{even}) \]

An index \( i \) is a channel index ⇔ either \( b_i b_{i+1} < 0 \) or \( b_i b_{i+1} > 4 \)

\( i \) is not a channel index ⇔ \( b_i b_{i+1} \geq 0 \) and \( b_i b_{i+1} \leq 4 \)

\( \iff (b_i, b_{i+1}) = (2, 2) \text{ or } (-2, -2) \)
Find an allowable path

Remark
For each of $[2, 4, 2]$ and $[-2, -4, -2]$, although $\exists 2$ channel indices, we can only find a path with 1 channel.

For $[2, b, 2]$ or $[-2, -b, -2]$ ($b \geq 6$), we have a path with 2 channels.
Find an allowable path

$L_{p/q}$: a hyperbolic 2-bridge link \( p/q = [b_1, \ldots, b_k] \) \((b_i \text{ : even})\)

We may assume either \( b_1 \geq 4 \) or \( b_1 = 2 \) and \( b_2 \leq -2 \).

Claim 1.

\([b_1, \ldots, b_k]\) contains 3 channel indices

\(\Rightarrow\) \(\exists\) an allowable path for \( p/q \) with 3 channels except for the cases

\([b_1, \ldots, b_k] = [b_1, 2, \ldots, 2, 4, 2, \ldots, 2]\) or
\([b_1, \ldots, b_k] = [b_1, -2, \ldots, -2, -4, -2, \ldots, -2]\).

In addition, each of them is expressed by one of the following:

1. \([2m + 1, 2n, 2, 2l - 1]\) with \( m \geq 1, n \leq -1, l \leq -1 \).
2. \([2m + 1, 2n - 1, 2, 2l]\) with \( m \geq 1, n \leq -1, l \leq -1 \).
3. \([2m + 1, 2n, -2, 2l - 1]\) with \( m \geq 1, n \geq 1, l \geq 2 \).
4. \([2m + 1, 2n - 1, -2, 2l]\) with \( m \geq 1, n \geq 2, l \geq 1 \).
Find an allowable path

Claim 2.

Suppose that $\frac{p}{q} = [b_1, \ldots, b_k]$ ($b_i : \text{even}, \ k \geq 3$) has at most 2 channel indices. Then $\frac{p}{q}$ can be expressed by one of the following:

1. $[2m + 1, 2n - 1]$ with $m \geq 1, \ n \neq 0, 1$
2. $[2m, 2n, 2l]$ with $m \geq 1, \ |n| \geq 2, \ |l| \geq 2$.
3. $[2m, 2n - 1, -2l]$ with $m \geq 1, \ |n| \geq 2, \ l \geq 1$.
4. $[2m, 2n + 1, 2l]$ with $m \geq 1, \ |n| \geq 2, \ l \geq 1$.
5. $[2m + 1, 2n, 2l - 1]$ with $m \geq 1, \ n \neq 0, \ l \neq 0, 1$.
6. $[2m + 1, 2n, -2, 2l - 1]$ with $m \geq 1, \ n \leq -1, \ l \geq 2$.
7. $[2m + 1, 2n - 1, -2, 2l]$ with $m \geq 1, \ n \leq -1, \ l \geq 1$.
8. $[2m + 1, 2n, 2, 2l - 1]$ with $m \geq 1, \ n \geq 1, \ l \leq -1$.
9. $[2m + 1, 2n - 1, 2, 2l]$ with $m \geq 1, \ n \geq 2, \ l \leq -1$. 
Proof of Theorem A

Combining the continued fractions listed in Claims 1 & 2, and using Lemma 3, we have the following.

**Theorem A. (again) [Ichihara-J.-Masai]**

$L$: a hyperbolic 2-bridge link

$L$ admits a complete exc. surg. \( \Rightarrow \) \( L \cong \) one of the following:

(a-1) \( L_{[2m+1,2n-1]} \) with \( m \geq 1, \ n \neq 0, 1. \)

(b-1) \( L_{[2m,2n,2l]} \) with \( m \geq 1, \ |n| \geq 2, \ |l| \geq 2. \)

(b-2) \( L_{[2m,2n-1,-2l]} \) with \( m \geq 1, \ |n| \geq 2, \ l \geq 1. \)

(b-3) \( L_{[2m,2n+1,2l]} \) with \( m \geq 1, \ |n| \geq 2, \ l \geq 1. \)

(b-4) \( L_{[2m+1,2n,2l-1]} \) with \( m \geq 1, \ n \neq 0, \ l \neq 0, 1. \)

(c-1) \( L_{[2m+1,2n,-2\text{sgn}(l),2l-1]} \) with \( m \geq 1, \ n \neq 0, \ l \neq 0, 1. \)

(c-2) \( L_{[2m+1,2n-1,-2\text{sgn}(l),2l]} \) with \( m \geq 1, \ n \neq 0, 1, \ l \neq 0. \)
Position of $L_{p/q}$

$S_r$: the 2-sphere with the center $O$ and radius $r$

$B = \bigcup_{0 < r \leq 2} S_r \cup \{O\}$: the 3-ball with radius 2

$S^3 = B \cup B'$ ($B'$: a 3-ball)

Set $L_{p/q} \subset S^3$ as:

$$L_{p/q} \cap S_r = \begin{cases} 
\emptyset & (0 < r < 1) \\
p/q\text{-arcs} & (r = 1) \\
4 \text{ points} & (1 < r < 2) \\
1/0\text{-arcs} & (r = 2)
\end{cases}$$

$r = 1$

$1 < r < 2$

$r = 2$
A part of $\Sigma_{p/q}$
For $r \geq 2 - \varepsilon$ part, take the following branched surfaces:
For $r \geq 2 - \varepsilon$ part, take the following branched surfaces:

For $r \leq 1 + \varepsilon$ part, take mirror of them and twists for $p/q$-arcs.
Slices corresponding to the edge from $1/0$ to $0/1$
For two vertices $p/q$ and $r/s$ with $|ps - qr| = 1$, we can construct such a branched surface in the same way.
Slices of a channel from $1/1$ to $1/3$
Slices of a channel from $1/1$ to $1/3$