On cosmetic surgery on knots

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Based on Joint work with

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and Zhongtao Wu (CUHK)

Papers

- (with Toshio Saito)
  Cosmetic surgery and the $SL(2, \mathbb{C})$ Casson invariant for 2-bridge knots.

- (with In Dae Jong (Appendix by Hidetoshi Masai))
  Cosmetic banding on knots and links.

- (with Zhongtao Wu)
  A note on Jones polynomial and cosmetic surgery.

- (with Toshio Saito and Tetsuya Ito)
  Chirally cosmetic surgeries and Casson invariants.

- (with In Dae Jong, Thomas W. Mattman, Toshio Saito)
  Two-bridge knots admit no purely cosmetic surgeries.
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Classification of 3-manifolds

FACT

Every closed orientable 3-manifold is;

- Reducible (containing essential sphere)
- Toroidal (containing essential torus)
- Seifert fibered (admitting a foliation by circles)
- Hyperbolic (admitting Riem. metric of const. curv. \(-1\))

as a consequence of the Geometrization Conjecture

including famous Poincaré Conjecture (1904)

conjectured by Thurston (late ’70s)

established by Perelman (2002-03)
What’s NEXT?

- Attack the remaining Open Problems.
  (e.g., “Heegaard genus VS rank of $\pi_1$” problem, ...)

- Relate Geometric & Topological invariants.  (e.g., Volume conjecture ...)

- .......

- Study the Relationships between 3-manifolds.  (e.g., Dehn surgery ...)  
  (↑ Today!)
### Dehn surgery on a knot

**K**: a knot (i.e., embedded circle) in a 3-manifold **M**

#### Dehn surgery on **K**

1) remove the open neighborhood of **K** from **M** (to obtain the exterior **E(K)** of **K**)
2) glue a solid torus back (along a slope **γ**)

We denote the obtained manifold by **K(γ)**.

A slope := an isotopy class of unoriented, non-trivial simple closed curve on **T²**

Slopes are parametrized by **\( \mathbb{Q} \cup \{ 1/0 \} \)** w.r.t. a meridian-longitude system.
Cosmetic surgery

It is natural to ask:

Can a pair of distinct Dehn surgeries give the same manifold?

Cosmetic Surgery Conjecture [Gordon, ’90], [Kirby’s list, Problem 1.81(A)]

A pair of Dehn surgeries on inequivalent slopes are never purely cosmetic.

▶ A pair of slopes are equivalent if there exists a homeomorphism of $E(K)$ taking one slope to the other.

▶ A pair of surgeries on $K$ along slopes $r_1, r_2$ are purely cosmetic if there exists an ori.-pres. homeo. between $K(r_1) \& K(r_2)$, and chirally cosmetic if the homeo. is orientation-reversing.
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Known facts (Examples)

Mathieu (1992)

The $\frac{18k+9}{3k+1}$- and $\frac{18k+9}{3k+2}$-surgeries on the trefoil knot $T_{2,3}$ in $S^3$ yield orientation-reversingly homeomorphic pairs for any $k \geq 0$.

Rong (1995)

Seifert knots in closed 3-manifolds (except lens spaces) admitting cosmetic surgeries are classified.

As a corollary, we have the following:
Let $T_{r,s}$ be the $(r, s)$-torus knot in $S^3$. For a positive integer $m$,

$$T_{r,s}(\frac{2r^2(2m+1)}{r(2m+1)+1}) \cong -T_{r,s}(\frac{2r^2(2m+1)}{r(2m+1)-1})$$

if and only if $r \geq 3$, odd, $s = 2$.

Matignion (2010)

Non-hyperbolic knots in lens spaces admitting cosmetic surgeries are classified.

Remark: These surgeries are all chirally cosmetic.
There exists a hyperbolic knot which admits a pair of surgeries along inequivalent slopes yielding oppositely oriented lens spaces.

The lens spaces are: $L(49, -19) \leftrightarrow L(49, -18)$ (mirror image)
Known facts (Criteria)

Let $\Delta_K(t)$ denote the Alexander polynomial of a knot $K$ in $S^3$ normalized to be symmetric and satisfy $\Delta_K(1) = 1$.

Boyer-Lines (1990)

A knot $K$ satisfying $\Delta''_K(1) \neq 0$ has no cosmetic surgeries.

They use the Casson invariant (defined by using $SU(2)$-representations).

Remark: $\Delta''_K(1) = \frac{1}{2} a_2(K)$ (the 2nd coefficient of the Conway polynomial of $K$)
Known facts (Criteria)

Based on [Heegaard Floer homology theory], developed by Ozsváth and Szabó, the following are obtained.

Theorem [Ni-Wu (2011)]

Suppose $K$ is a nontrivial knot in $S^3$ & $r_1, r_2$ are distinct slopes such that $K(r_1) \cong K(r_2)$ as oriented manifolds.

Then $r_1, r_2$ satisfy that

(a) $r_1 = -r_2$
(b) suppose $r_1 = p/q$, then $q^2 \equiv -1 \pmod p$
(c) $\tau(K) = 0$, where $\tau$ is the invariant defined by Ozsváth-Szabó.

They use Heegaard Floer $d$-invariant and $\tau$-invariant.

Remark: If $K$ is alternating, $\tau(K) = \sigma(K)$ (signature of $K$)
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Jones polynomial

Next theorem gives a severe restriction for a knot in $S^3$ to admit purely cosmetic surgeries in terms of its Jones polynomial.

Theorem [I.-Wu, 2019]

Let $V_K(t)$ be the Jones polynomial of a knot $K$ in $S^3$. If a knot $K$ satisfies either $V''_K(1) \neq 0$ or $V'''_K(1) \neq 0$, then $K(r) \not\cong K(r')$ as oriented mfds. for distinct slopes $r$ and $r'$.

Remark

Boyer and Lines obtained a similar result for a knot $K$ with $\Delta''_K(1) \neq 0$ by using the Casson invariant.

$$\Delta_K(t): \text{the normalized Alexander polynomial}$$

Since $V''_K(1) = -3\Delta''_K(1)$, our result can be viewed as an extension of [Proposition 5.1, Boyer-Lines (1990)].
Lescop’s $\lambda_2$ invariant

The essential new ingredient for our result is the following.

Lescop’s $\lambda_2$ invariant (2009)

The invariant $\lambda_2 := W_2 \circ Z_2$, where $W_2$ is a linear form on $\mathcal{A}_n$

with $W_2(\begin{array}{c} \otimes \end{array}) = 1$ and $W_2(\begin{array}{c} \otimes \otimes \end{array}) = 0$.

$\mathcal{A}_n$: the vector space generated by degree $n$ Jacobi diagrams subject to AS and IHX relations

$Z_n$: the degree $n$ part of the Kontsevich-Kuperberg-Thurston invariant of rational homology spheres taking its value in $\mathcal{A}_n$

Remark: KKT-inv. is a universal finite type invariant for integral homology spheres (in the sense of Ohtsuki, Habiro and Goussarov)
From $\lambda_2$ to $w_3$

**Fact [Theorem 7.1, Lescop (2009)]**

The invariant $\lambda_2$ satisfies the surgery formula

$$\lambda_2(K(p/q)) = (\frac{q}{p})^2 \lambda_2''(K) + (\frac{q}{p}) w_3(K) + c(\frac{q}{p}) a_2(K) + \lambda_2(L(p, q))$$

for all knots $K \subset S^3$.

- $a_2(K)$: the $z^2$-coefficient of the Conway polynomial $\nabla_K(z)$
- $L(p, q)$: the lens space (obtained by $p/q$ surgery on the unknot)
- $\lambda_2''(K) \& c(\frac{q}{p})$: explicit constants defined in [Lescop, 2009]

$w_3(K)$ is a knot invariant defined by Lescop.
The invariant $w_3$

Key Lemma

For all knots $K \subset S^3$,

$$w_3(K) = \frac{1}{72} V''_K(1) + \frac{1}{24} V'_K(1).$$

This can be shown in the same line as [Prop. 4.2, Nikkuni (2005)] by using the skein relation for $w_3$ given by Lescop.

Remark:

For a knot $K \subset S^3$, $V''_K(1) = -6a_2(K) = -3\Delta''_K(1)$. 
Ozsváth and Szabó gave the example of $K = 9_{44}$, which is a genus two knot such that $K(1)$ and $K(-1)$ have the same Heegaard Floer homology. On the other hand, our theorem can detect that they are not homeomorphic.

**Corollary**

The cosmetic surgery conjecture is true for all knots with no more than 11 crossings, except possibly

$$10_{33}, 10_{118}, 10_{146}, 11a_{91}, 11a_{138}, 11a_{285}, 11n_{86}, 11n_{157}.$$  

Recently, these results are extended by T. Ito by using LMO invariant.

**Remark:** LMO-inv. is another universal finite type invariant for $\mathbb{Z}$HS.
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2-bridge knot

A knot admitting a diagram with two maxima and minima.

Denote this diagram by $C[2, 2, -2, 2, 2, -2]$ (Conway form).

Also a 2-bridge knot is parametrized by the slope $p/q$ (irreducible fraction).

Denote it by $S(p, q)$ (Schubert form).

Fact

For a 2-bridge knot with $S(p, q)$ & $C[a_1, \ldots, a_n]$, $p/q = [a_1, a_2, \ldots, a_n]$ (continued fraction expansion)
Result


Two-bridge knots admit no purely cosmetic surgeries.

Also, we have the following.

Theorem.

All alternating fibered knots and all alternating pretzel knots admit no purely cosmetic surgeries.
Hanselman’s result (Heegaard Floer theory)

Theorem [Hanselman, arXiv:1906.06773]

Let $K$ be a knot in $S^3$. Suppose that $K(r) \cong K(r')$ for distinct $r, r'$. Then the following holds.

(i) $\{r, r'\} = \{\pm 2\}$ or $\{\pm 1/q\}$,
(ii) if $\{r, r'\} = \{\pm 2\}$, then $g(K) = 2$, ($g(K)$ denotes the genus of $K$)
(iii) if $\{r, r'\} = \{\pm 1/q\}$, then

$$q \leq \frac{th(K) + 2g(K)}{2g(K)(g(K) - 1)}$$

(th($K$) denotes the thickness of Heegaard Floer homology)

Lemma

Let $K$ be an alternating knot in $S^3$. If $K$ admits purely cosmetic surgeries, then $g(K) = 2$, the signature $\sigma(K) = 0$, and the surgery slopes must be $\pm 1$ or $\pm 2$. 
By using Jones polynomial of knots, we have the following.

Lemma ([I.-Wu, CAG, 2019])

If a 2-bridge knot of genus two admitting purely cosmetic surgeries, then it would be associated to the continued fraction \([2x, 2y, -2(x + y), 2x]\) for integers \(x > 0\) and \(y \neq 0\).
Let $K$ be a 2-bridge knot associated to the continued fraction $[2x, 2y, -2(x + y), 2x]$ for integers $x > 0$ and $y \neq 0$.

**Proposition 1.**

If $K$ admits purely cosmetic surgeries, then $y < 0$ and $(x + y) > 0$.

We use the following result of [Lee] and [Traczyk]:

for a reduced alternating diagram $D$ of an oriented non-split alternating link $L$,

$$\sigma(L) = o(D) - y(D) - 1$$

It remains to handle the case of $y < 0$ and $(x + y) > 0$.

In this case, the simple (positive) continued fraction for $K$ is $[2x - 1, 1, -(2y + 1), 2(x + y) - 1, 1, 2x - 1]$. 

**Signature** $\sigma(K)$
Let $K$ be a 2-bridge knot associated to the continued fraction $[2x - 1, 1, -(2y + 1), 2(x + y) - 1, 1, 2x - 1]$ for $x > 0$, $y < 0$ with $(x + y) > 0$.

**Proposition 2.**

If $K$ admits purely cosmetic surgeries, then $x = -2y$.

Note that the knot is amphicheiral when $x = -2y$.

Our key ingredient is the $SL(2, \mathbb{C})$ Casson invariant, originally by [Curtis]. A practical surgery formula for 2-bridge knots was obtained by [Boden-Curtis], and was used for a study of cosmetic surgeries on 2-bridge knots in [I.-Saito].
Finite type invariants

Note that if $x = -2y$, then the knot $K$ is associated to the continued fraction $[4n, -2n, -2n, 4n]$ for $n > 0$.

Proposition 3.

The 2-bridge knot $K$ associated to the continued fraction $[4n, -2n, -2n, 4n]$ for a positive integer $n$ admits no purely cosmetic surgeries.

We use the obstructions obtained by [Boyer-Lines] and [Ito]:
If a knot $K$ has a purely cosmetic pair of surgeries, then

- $a_2(K) = 0$
- $j_4(K) \neq 14n^4$ and $j_4(K) \neq 284n^4$ for some $n > 0$.

On the other hand, by direct calculations, we have

$$\nabla_K(z) = 1 + 4n^4z^4 \quad \text{and} \quad j_4(K) = -12n^4$$

for $K = C[4n, -2n, -2n, 4n]$ with $n > 0$. 

$\square$
Recent progress

The purely cosmetic surgery conjecture is true for the Kinoshita-Terasaka and Conway knot families
Purely cosmetic surgeries and pretzel knots
[arXiv:2005.12795] Ina Petkova, Biji Wong
Twisted Mazur pattern satellite knots and bordered Floer theory
3-braid knots do not admit purely cosmetic surgeries
Connected sums of knots do not admit purely cosmetic surgeries
Two-bridge knots admit no purely cosmetic surgeries (to appear in Algebr.Geom.Topol.)
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Known examples

Recall: A pair of surgeries on $K$ along slopes $r_1, r_2$ are chirally cosmetic if there exists an orientation-reversing homeo. between $K(r_1) \& K(r_2)$.

<table>
<thead>
<tr>
<th>amphicheiral knot</th>
</tr>
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<tbody>
<tr>
<td>When $K$ is amphicheiral knot, for all slope $r \not\in {0, 1/0}$, $r$- &amp; $(-r)$-surgeries are chirally cosmetic along equivalent slopes.</td>
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</tbody>
</table>

Mathieu, 1992

The $(18k + 9)/(3k + 1)$- and $(18k + 9)/(3k + 2)$-surgeries on the trefoil knot $T_{2,3}$ in $S^3$ are chirally cosmetic for any $k \geq 0$.

Remark: The pair of slopes are inequivalent.
Alternating knot of genus one

Theorem [I.-Ito-Saito]

Let $K$ be an alternating knot of genus one. For distinct slopes $r$ and $r'$, if the $r$- and $r'$-surgeries on $K$ are chirally cosmetic, then either

(i) $K$ is amphicheiriral and $r = -r'$, or

(ii) $K$ is the right-handed trefoil and $\{r, r'\} = \left\{\frac{18k+9}{3k+1}, \frac{18k+9}{3k+2}\right\}$, or, $K$ is the left-handed trefoil and $\{r, r'\} = \left\{-\frac{18k+9}{3k+1}, -\frac{18k+9}{3k+2}\right\}$ ($k \in \mathbb{Z}$).

**Ingredients:** $SL(2, C)$-Casson inv. & finite type invariants

**Question.**

Can a non-torus, chiral knot in $S^3$ admit chirally cosmetic surgeries?
Theorem [I.-Jong, 2019]

There exists a hyperbolic knot with chirally cosmetic surgeries along inequivalent slopes yielding **hyperbolic** manifolds.

This gives a **counter-example** to another conjecture:

**Conjecture [Bleiler-Hodgson-Weeks, [Kirby’s list Problem 1.81(B)]]**

Any hyperbolic knot admits no purely/chirally cosmetic surgeries yielding hyperbolic manifolds.
Montesinos trick & Banding

Montesinos trick [Montesinos, 1975]

Let $\tilde{M}$ be a 3-mfd with a double br.cover $\tilde{M}$ along a link $L \subset M$. Suppose that a knot $K$ in $\tilde{M}$ is strongly invertible w.r.t. the axis $\tilde{L}$.

Then, for a slope $r \in \mathbb{Z}$, the mfd. $K(r)$ is homeomorphic to the double br.cover along the link obtained from $L$ by a banding.
Bleiler-Hodgson-Weeks’s Example

Bleiler-Hodgson-Weeks’s example (1999); yielding $L(49, -19)$.

\[ \downarrow \text{(Cosmetic banding on the knot } 9_{27} \text{)} \]
Bleiler-Hodgson-Weeks’s Example

mirroring & \(2\pi/3\)-rot.

banding of slope 0
Construction

- This knot $K$ admits a (chirally) cosmetic banding.
- The double branched cover $M$ along $K$ is hyperbolic.
- The knot $\overline{K}$ in $M$ corresponding to the banding is hyperbolic.
- The knot $\overline{K}$ is not amphicheiral.
How to check?

- The double branched cover $M$ along $K$ is hyperbolic.
- The knot $\tilde{K}$ in $M$ corresponding to the banding is hyperbolic.  
  \[\text{checked by HIKMOT}\]

- The knot $\tilde{K}$ is not amphicheiral.  
  \[\text{checked by HIKMOT} + \text{[Dunfield-Hoffman-Licata]}\]

References:


Thank you for your attention!