Pairs of boundary slopes with small differences

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Alternating knot
Montesinos knot

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Proof

Surface 1
Surface 2
Essential surface

\[ K : \text{a knot in } S^3 \]

\[ E(K) : \text{the exterior of } K, \]
i.e., \( E(K) = S^3 - \text{open tubular nbd. of } K \)
Essential surface

$K$: a knot in $S^3$

$E(K)$: the **exterior** of $K$, i.e., $E(K) = S^3$ – open tubular nbd. of $K$

**Essential surface**

An embedded surface $F$ in $E(K)$ is called **essential** if $F$ is incompressible and $\partial$-incompressible.

**Remark:**
Surfaces are not assumed to be orientable.
**Boundary slope**

**Slope**

A slope on $\partial E(K)$ means the isotopy class of non-trivial simple closed curves on $\partial E(K)$. 

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Slope

A slope on $\partial E(K)$ means the isotopy class of non-trivial simple closed curves on $\partial E(K)$.

Fix standard meridian-longitude system for $K$.

Then $\{ \text{slope on } \partial E(K) \} \leftrightarrow 1:1 \ \mathbb{Q} \cup \{ \frac{1}{0} \}$
Boundary slope

A slope on $\partial E(K)$ means the isotopy class of non-trivial simple closed curves on $\partial E(K)$.

Fix standard meridian-longitude system for $K$.

Then \[ \{ \text{slope on } \partial E(K) \} \xleftrightarrow{1:1} \mathbb{Q} \cup \{ \frac{1}{0} \} \]

Let $F$ be an essential surface in $E(K)$.

The boundary slope of $F$ is defined as the slope determined by boundary components of $F$. 
**Examples**

**Example 1. (figure-eight knot) [Thurston]**

```
+4
```

```
−4
```

boundary slopes are $−4, 0, 4$

“0” indicates the boundary slope of Seifert surfaces.

- (Hatcher-Thurston) For 2-bridge knots, there exists an algorithm to determine all boundary slopes.
Examples

Example 2.

\[ K = 8_{20} \]: non-alternating knot

boundary slopes are \(-10, 0, \frac{8}{3}\).
Examples

Example 2.

\[ K = 8_{20} : \text{non-alternating knot} \]

boundary slopes are \(-10, 0, \frac{8}{3}\).

Example 3. [Tsau]

For a Torus knot \( T_{p,q} \), boundary slopes are \( 0, pq \).

“0” : the boundary slope of a Seifert surface

“pq” : the boundary slope of an essential annulus
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Culler-Shalen’s results

Culler-Shalen (1984)

Every non-trivial knot has **at least two** boundary slopes.
**Culler-Shalen’s results**

**Culler-Shalen (1984)**

Every non-trivial knot has **at least two** boundary slopes.

**Culler-Shalen (1999)**

Any non-trivial knot in $S^3$ not having the meridional boundary slope admits a pair of boundary slopes whose difference is at least 2.

**Remark**

“difference” means the difference **as rational numbers**.
Any alternating knot admits a pair of boundary slopes whose difference is at least twice of its crossing number.

Also the difference is bounded from below in terms of minimal genera of bounding surfaces.

The boundary slopes are given by the pair of checker-board surfaces for a reduce alternating diagram.
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The boundary slopes are given by the pair of checker-board surfaces for a reduce alternating diagram.

Similar bounds for Montesinos knots were obtained, and reported in past 東北結び目セミナー.
Montesinos knot

\[ M\left( \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n} \right) \]

Assume that:

- the number of tangles \( n \) is at least 3,
- all fractions are non-integral.
Montesinos knot

**Theorem [I.-Mizushima, 2009]**

Let $K$ be a non-trivial Montesinos knots, and $Cr(K)$ its minimal crossing number. Then $K$ has a pair of boundary slopes whose difference is at least $Cr(K) - 6$.

(東北結び目セミナー -2006 in Late Autumn-)
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Main Theorem

Theorem (I., preprint)

For any positive number $\varepsilon$, there exists a knot in $S^3$ admitting a pair of boundary slopes whose difference is at most $\varepsilon$.

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Main Theorem

**Theorem (I., preprint)**

The Montesinos knot \( K_n = M(-1/2, 2/5, 1/n) \) with an odd positive integer \( n \geq 11 \) admits a pair of boundary slopes \( 2(n - 1)^2/n \) and \( 2(n^2 - 9n + 15)/(n - 7) \).

**Note:**

\[
\frac{2(n^2 - 9n + 15)}{n - 7} - \frac{2(n - 1)^2}{n} = 2 \left( \frac{1}{n} + \frac{1}{n - 7} \right) \to 0
\]

as \( n \to \infty \).
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**Algorithm by Hatcher and Oertel**

: finds and enumerates all boundary slopes for a given Montesinos knot.

: is based on Algorithm by Hatcher and Thurston for two-bridge knots.

: seems to be algebraic or combinatorial.

(i) produces candidate surfaces.

(ii) verifies incompressibility of surfaces.

**Ref.**

A. Hatcher and U. Oertel,
Boundary slopes for Montesinos knots,
Dunfield’s software

- implements Algorithm of Hatcher and Oertel.
- is available from
  http://www.math.uiuc.edu/~nmd//montesinos/index.html
- is written in Python language.

Computer experiments have been very helpful !!
Edge path system

In algorithm,

\[
\text{essential surface} \uparrow
\]

system of sequences of irreducible fractions
(we call an edge path system)

Example. A Seifert surface for \( K(-1/2, 1/3, 1/7) \)

\[
\begin{align*}
\langle \infty \rangle & - \langle \frac{0}{1} \rangle - \langle -\frac{1}{2} \rangle \\
\langle \infty \rangle & - \langle \frac{1}{1} \rangle - \langle \frac{1}{2} \rangle - \langle \frac{1}{3} \rangle \\
\langle \infty \rangle & - \langle 1 \rangle - \langle \frac{1}{2} \rangle - \langle \frac{1}{3} \rangle - \langle \frac{1}{4} \rangle - \langle \frac{1}{5} \rangle - \langle \frac{1}{6} \rangle - \langle \frac{1}{7} \rangle
\end{align*}
\]
Surface $\Rightarrow$ edge path system

Decompose $S^3$ into balls $B_1, \cdots, B_n$. $B_i \supset T_i$
Subsurfaces

An essential surface $F$ can be isotoped to be in a **standard position** in each $B_i$:

- (a) base disks
- (b) saddle
- (c) compound
- (d) cap
In $B_i \supset S^2 \times [0, 1]$, the intersection curves of $F$ and level spheres give an edge path system.
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Edge path system & saddle surface

\[ \langle \frac{p}{q} \rangle - \langle \frac{r}{s} \rangle \iff |p \cdot s - q \cdot r| = 1 \]
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**Edgepath system 1**

\[ \Gamma = (\gamma_1, \gamma_2, \gamma_3) \]

\[ \gamma_1 : \left( \frac{1}{n} \langle -1 \rangle + \frac{n - 1}{n} \langle -\frac{1}{2} \rangle \right) - \langle -\frac{1}{2} \rangle \]

\[ \gamma_2 : \left( \frac{1}{n} \langle 0 \rangle + \frac{n - 1}{n} \frac{1}{2} \langle \frac{1}{2} \rangle \right) - \frac{1}{2} - \frac{2}{5} \]

\[ \gamma_3 : \left( \frac{n - 1}{n} \langle 0 \rangle + \frac{1}{n} \langle \frac{1}{n} \rangle \right) - \frac{1}{n} \]

---

**Note:**

A point like \( \frac{p}{q} \) corresponds to (c) compound
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Edgepath system 2

\[ \Gamma' = (\gamma'_1, \gamma'_2, \gamma'_3) \]

\[ \gamma'_1 : \frac{n - 7}{n - 4} \langle -\frac{1}{2} \rangle + \frac{3}{n - 4} \langle -\frac{1}{2} \rangle^\circ \]

\[ \gamma'_2 : \left( \frac{n - 9}{n - 7} \langle \frac{1}{2} \rangle + \frac{2}{n - 7} \langle \frac{2}{5} \rangle \right) - \langle \frac{2}{5} \rangle \]

\[ \gamma'_3 : \left( \frac{n - 8}{n - 7} \langle 0 \rangle + \frac{1}{n - 7} \langle \frac{1}{n} \rangle \right) - \langle \frac{1}{n} \rangle \]

Note:

A point like \( k \langle p/q \rangle + l \langle p/q \rangle^\circ \) corresponds to \( (d) \) cap.